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Fermi problems as a hub for task design in mathematics and stem education

Jonas Bergman Ärlebäck[†,*] and Lluís Albarracín[‡]
[†]*Department of Mathematics, Linköping Universitet, 581 83 Linköping, Sweden*
[‡]*Departament de Didàctica de la Matemàtica i les Ciències Experimentals, Universitat Autònoma de Barcelona, Campus de la UAB, Edifici G5, 08193 Bellaterra, Spain*

[*]*Corresponding author.* jonas.bergman.arleback@liu.se

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In this paper, we draw on recent research on so-called Fermi problems and situate the fundamental principles underlying this type of tasks and their use from a task design perspective. We use the models and modelling perspective on teaching and learning to elaborate on aspects related to the design of single-use, as well as sequences of, Fermi problems. In addition, we discuss a framework (called the Fermi problem activity template [FPAT] framework) for supporting the design and use of Fermi problems to facilitate students' learning within particular mathematics content areas and/or aimed at particular concepts or higher-order thinking skills. We also illustrate how the FPAT framework can be used to (i) facilitate interdisciplinary collaborations with other subjects such as the social sciences, but in particular with the other STEM subjects; and (ii) support teachers in adapting and implementing Fermi problems in their classrooms.

1. Introduction

Since the 1970s, there has been a steadily increasing interest within the field of mathematical education centered around the idea of bringing mathematics classroom activities closer to phenomena and experiences in the real world (Blum, 2002). In this context, mathematical modelling, as generally understood, is the use of mathematics to describe, predict, understand or explain real-world situations or phenomena (Blomhøj & Højgaard Jensen, 2003; Niss & Blum, 2020). However, studies have shown that although an increased emphasis in curricula on modelling, teachers have difficulties in adapting and implementing modelling tasks in their classrooms, especially dealing with the openness modelling tasks bring with respect to (1) the extra-mathematical knowledge needed; (2) how to tackle and productively use students' diverse ideas and strategies; and (3) that engaging in modelling is time consuming (Borromeo Ferri & Blum, 2013; Blum, 2015; Borromeo Ferri, 2021). General potential solutions and strategies for coming to terms with these three difficulties have been discussed to various degrees, and mostly separately, in the research on *task design* (Margolinas, 2013; Watson & Ohtani, 2015). Watson & Ohtani (2015) discuss three different *theoretical grain sizes* of frameworks and set of principles for task design: *grand*-, *intermediate*- and *domain-specific frames*. Grand frames are typically general learning

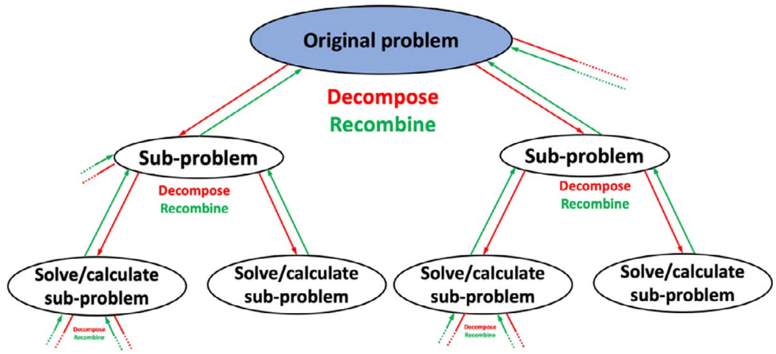

FIG. 1. The Fermi estimate method visualized.

theories and the implication of these how to best design tasks in line with the assumptions of the theory in question to support the learning of mathematics. Intermediate frames are more local in the sense that they acknowledge the situational complexity of leaning, and often involve interactions between task, teacher, teaching methods, educational environment and mathematical knowledge. Domain-specific frames, on the other hand, 'focus on particular areas of mathematical knowledge or activity and may not be generalizable across mathematics' (p. 6). In this paper, we want to connect to this line of research and discuss and highlight the potential of so-called *Fermi problems* (FPs) to collectively address the three challenges in terms of domain-specific- and intermediate design ideas.

1.1. Fermi problems

FPs can be considered being miniature-modelling problems in the sense that they are smaller, more well-defined and delimited real-world problems (Robinson, 2008). As such, they capture the essence of full modelling problems, yet do not exhibit the same complexity nor put the same level of demands on teachers or students. FPs were originally used by the physicist Enrico Fermi to illustrate the power of deductive thinking and to prepare his own and his students' experimental work in order to save time and work more effectively (Efthimiou & Llewellyn, 2006). The most prominent characteristic that defines FPs is the way in which their solutions are achieved. Fermi posed problems which at first glance seemed impossible to solve, but that could be tackled by making assumptions based on common knowledge, following simple chains of reasoning and engaging in simple calculations. This aspect of the modelling problem processes is what Edwards and Hamson (1996, p. 39–40) discuss in the development of mathematical models using *rough estimates*. The procedure proposed by Fermi was to decompose the original problem into simpler sub-problems to reach a solution of the original question by means of reasonable estimates or educated guesses (Carlson, 1997); see Fig. 1 below. In the literature, this way of working is known as the *Fermi (estimates) method*, and as Barahmeh *et al*. (2017) showed, it can be both appreciated by, and successfully taught to, students. According to Efthimiou & Llewellyn (2006), FPs are often sparsely worded and provide a limited amount of information, such as the following examples of FPs: *How many jelly beans can you fit in a one-litre bottle? How many pizzas are ordered in your city this year? How much gasoline is yearly consumed in our country?* Some FPs may pose seemingly banal questions but do contain and connect to relevant mathematical content and skills, and in addition, can pinpoint and help focus students on important social or environmental issues (Carlson, 1997), as well as foster critical mathematical thinking (Sriraman & Knott, 2009).

2. FPs in the context of the models and modelling perspective and model development sequences

In this paper, we situate FPs with respect to the *models and modelling perspective* (MMP), in which a model is defined as a general system consisting of elements, relationships, rules and operations that can be used to make sense of, predict, describe or explain some other system. A mathematical model specifically focuses on the structural characteristics of the system in question (Lesh & Doerr, 2003). From this perspective, learning is understood as developing useful and generalized models that are made up of (1) a set of concepts used to describe or explain the mathematical objects and aspects in context relevant to the phenomenon studied, and (2) procedures that can be used or re-used to create useful constructions, manipulations or predictions for achieving clearly recognized goals in a range of contexts (Lesh & Harel, 2003).

The elements of task design inherent in the MMP come to the fore in the three different types of structurally related activities organized in so-called *model development sequences* purposefully designed to support students' learning towards a given learning goal. In short, the three types of activities are: *model eliciting activities* (MEAs) which aim to elicit the students' ideas they bring the activity; *model exploration activities* (MXAs) that focus on the underling mathematical structure elicited by students; and *model application activities* (MAAs) were students apply their model in similar or new contexts (Lesh *et al.*, 2003). In all three types of activities students iteratively engage in expressing, testing, revising and developing their models (Lesh *et al.*, 2003; Lesh & Doerr, 2003).

2.1. *FPs conceived as MEAs*

Lesh *et al*. (2000) developed six design principles for MEAs that can be summarized as: (i) the activity must appear meaningful to the students; it should allow students to (ii) create and (iii) evaluate mathematical models, and the models should be (iv) effective but as simple as possible; (v) the students' work must be adequately documented; and the constructed models should be (vi) generalizable and useable in other situations. These six design principles have strong resemblance with the characteristics of FPs (Ärlebäck, 2009) and can support the designing of FP focused on eliciting and introducing specific mathematical content, concepts and procedures for further development, such as, for example, the distribution of objects in a surface (How many people would fit in the Chase Center Arena (San Francisco) during a concert?) or in a volume (How many coins would fit into a cubic safebox of side 1 m?), or the stratification of within a population (What is the sum of the ages of all the people in the school?) (Albarracín & Gorgorió, 2015). Situating FPs as MEAs highlight the potential of FPs function as vehicle for designing activities focusing on various subject matter content.

However, as MEAs, FPs can additionally be considered means in themselves that facilitate designs of activities that support students towards developing higher-order skills which are inherent in the Fermi method such as problem solving, modelling and critical thinking. For example, as noted in Ärlebäck & Albarracín (2019), previous studies FPs have been used successfully to introduce mathematical modelling in primary schools (Peter-Koop, 2009; Henze & Fritzlar, 2010; Haberzettl *et al*., 2018) and secondary schools (Ärlebäck, 2009; Albarracín & Gorgorió, 2014; Greefrath & Frenken, 2021), as well as at the undergraduate level (Czocher, 2016, 2018). Indeed, Borromeo Ferri (2018) stresses that FPs are useful to introduce modelling in classrooms since they allow students to develop problem-solving strategies based their own emerging questions about the real-life phenomena studied. When working at the primary level, the role of the teacher becomes important in pre-structuring the FP to facilitate that the students productively can engage in the Fermi method to elicit these higher-order skills (Albarracín, 2021). Another approach to support students' work on FPs is to use sequences of FP to both familiarize the students to the way of working as well as providing a learning goal aimed towards a particular curricular content.

2.2. Sequences of FPs conceived within model development sequences

From the MMP, and depending on the learning goal at hand, PFs can have different functions within a modelling development sequence. As pointed out above, FPs can quite naturally serve as MEAs. However, as the following example illustrates, FPs can also play the role as an MXA or as an MAA, or both. The potential in using a MEA, as emphasized by Ärlebäck & Doerr (2015), is that special attention can be given to the evolving learning space promoted by the MEA. In particular, this means that the activities that follow the MEA allow students to (1) refine or redefine the constructed models by contrasting the weaknesses and strengths of different types of representations; (2) using language in a precise manner; and (3) using representations purposely and productively (Ärlebäck & Doerr, 2015). In Albarracín & Gorgorió (2018), a modelling development sequence was designed exclusively using FPs. The leaning goal of the sequence was to introduce students to ideas related to density, and the tasks were designed around different contexts in which students were asked to estimate the number of objects in two-dimensional enclosed areas. The FPs used were the following:

- FP1: How many people could we fit in the high school courtyard?
- F2.1: How many people could we fit in the Palau St. Jordi sport pavilion for a concert?
- FP2.2: How many people could we fit in the town hall square during a public protest?
- FP2.3: How many people could we fit in Plaça Catalunya square during a public protest?
- FP3: How many trees are there in Central Park?

FP1 acts as a MEA in that it aims at eliciting students' initial ideas about how to solve the problem in the context of a situation that is highly relatable and accessible to the students. The proximity of the courtyard encourages the students to go outside to try out their ideas and explore to make informed decisions or take measurements. The problems FP2.1 to FP3 maintain the same principle objective as FP1, but change (i) the accessibility of the enclosed area in the task; (ii) the geometrical characteristics of the enclosure; and/or (iii) the way in which people are placed or occupy space. This encourages the students to reconsider whether the models developed in FP1 allow them to tackle the new situation, leading them to adapt and develop their models, or even revise them completely. In this sense, the FP2.1 to FP3 problems act as MXAs, encouraging the students to explore the mathematical structure of the elicited models in FP1. The last task in the sequence, FP3, function as a MAA in that it introduces variation in the objects being fitted (trees instead of people), which further promotes the students to readapt the concepts and procedures of their models used in the previous problems situations. The study by Albarracín & Gorgorió (2018) showed that students work in small groups on the sequence of FPs did promote the development and use of densities in terms of population density models.

A selection of the sequence of problems listed above (FP1, FP2.2, FP2.3 and FP3) was also used in the study by Albarracín *et al*. (2022) with a slightly modification to the two locations mentioned in the FPs corresponding to FP2.2 and FP2.3. Drawing on a *downscaling—upscaling task design framing* of the sequences of tasks (c.f. Pla-Castells & Ferrando, 2019), and an experimental design using a treatment and a control group, Albarracín *et al*. (2021) found that (i) the experimental group working through the whole sequence of FPs were better equipped to tackle FP3 than the control group (which only worked on the FP3 problem); and (ii) that the experimental groups' strategies were more homogenous with respect to the underlying idea (using iteration of a base unit) compared to the strategies employed by the control group. These two results show that by working through a well-designed sequence of

FPs, it is possible for students to successively be eased into increasingly more unfamiliar contexts entailing gradually more complex aspects that potentially could be in-cooperate in their solutions, and that this can be done in such a way that the students develop powerful models and skills to apply them effectively.

3. Fermi problem activity template—A tool for (interdisciplinary) task design using FPs

In our review of the literature on FPs in different disciplines (Ärlebäck & Albarracín, 2019), we discussed FPs as integrators *between* the STEM disciplines; as facilitators for learning *in* the STEM disciplines; and their connections to learning 21st century skills (Binkley *et al*., 2012). In addition, doing the review allowed us to identify four types of mathematical activities (c.f. Rasmussen *et al*., 2005) that are broadly used to achieve the numerical values needed of quantities to be able to provide a solution and answer the problems in question. We have termed these four activities *guesstimation* (the *estimation* in 'standard' FP solving), *experimentation*, *looking for data* and *polling or statistical data collection*. Below we briefly describe and provide examples of these four activities, discuss how these align with the problem solving process of FPs and highlight the natural feature of FP to facilitate interdisciplinary collaboration in term of references from other fields.

Guesstimation. An original feature of FPs is the answering of a question by combining educated guessed estimates to sub-problems. Generally, estimation is a process that gives a rough solution to a problem in counting or measurement. Shakerin (2006), working in a STEM setting, argues that part of an engineer's practice is to use estimation to solve ill-defined problems or when detailed solutions are not required, which directly connects with FPs. In STEM, estimation particularly plays an important role in (a) preliminary stages of design processes; (b) decision-making based on incomplete or unavailable details/data; (c) forced selections made from a multitude of options affecting outcome variables; and (d) in checking the validity of decisions and calculations (Shakerin, 2006).

Experimentation. Developing laboratory skills in planning and doing physical experiments was one of Fermi's original motivations for using his method. Practical experiences and purposeful physical experimentation can result in adequate data for the quantities considered relevant to solve a given problem. In many cases, this experimentation can be carried out outside the classroom and linked to for example counting or measuring processes. Gómez & Albarracín (2017) showcase this in the work of Year 1 students, engaged in a practical activity outside the school, estimating the number of people living in their city by approximating the number of houses or flats in the students' own street. On the other hand, Bentley (1984), when computer storage and computer speed were considerably more limited than today, showcased and argued that the using *computer experimentation* (running and clocking small and simple snippets of code) can productively inform Fermi method based on calculations to estimate the running time of computer programs and the need of memory. A similar argument like the one put forward by Bentley can be applied today in the context of questions regarding for example the dimensioning of artificial intelligent system, such as with respect to the number of layers and parameters in the design of a machine learning systems.

Looking for data. In many cases, some of the quantitative data needed in solving a FP can be found by consulting external records and sources, such as Wikipedia or national statistical institutes. This could be the case when for example the population of a city or a country is needed. However, in other cases identifying reliable sources might not be so easy, as they potentially contain errors or biases induced for unknown reasons. In this case, the work with FPs can function as a tool for critically evaluating the information to either repudiate or validate the public sources and published data. In biology, Phillips & Milo (2009) initiated the project www.bionumbers.org which collects experimentally reliable and validated values of quantities relevant for research in biology.

Polling or statistical data collection. Sriraman & Knott (2009) suggest FPs to have the potential to lead to a growing awareness of ecological and environmental problems, as well as to provoke a critical stance towards governmental and corporate policies regarding freshwater consumption, gasoline consumption, wastage of food, amount of trash produced, etc. One way to get trustworthy values for relevant quantities in such questions, is to make estimates and subsequent checks with official data, but it is also possible to engage in data collection and statistical analysis in the classroom. In this latter case, the students can for example create surveys and decide on appropriate samples to explore and investigate various problems related to societal issues. An example of this approach is provided by Blomberg (2015) how studied upper secondary students learning statistic in a designed sequence of lessons working on the question 'What proportion of Swedish youth walk at least 10,000 steps a day?'. The students first approached this question as a FP developing hypotheses which they tested by collecting data using pedometers and engaging in statistical analyses.

In Albarracín & Ärlebäck (2019) we pull these four different activities together to a framework for analysis and design called Fermi problem activity templates (FPATs).

4. FPs ACTIVITY TEMPLATES

An FPAT is a characterization that focuses on the structure of the FP in terms of the four different types of activities outlined above. FPATs can, for example, be used (i) as a tool for designing and/or doing *a priori* analyses of FPs given particular learning goals; or (ii) for analyzing the work of students unguided work on modelling problems in general, and on FPs in particular. In terms of using FPATs as a tool for task design, a FPAT is especially helpful to the tackle some the difficulties related to the openness of FPs, and to realize and anticipate possible ways to divide the original problem into potential interconnected subproblems needed to be solved—and how these are related to the learning goal and other mathematical or STEM disciplinary content. The focus on different mathematical activities of the FPAT is an aspect of the framework that can be described as providing an *intermediate frame* for task design (cf. Watson & Ohtani, 2015) in that its four activities (*guesstimation*, *experimentation*, *looking for data* and *polling or statistical data collection*) are general problems-solving strategies applicable to many mathematical areas (as well as applied disciplines). Further, the FPAT framework used as a design tool facilitate to think about and acknowledge the situational complexity of leaning, and often involve such aspects as the interactions between task, teacher, teaching methods, educational environment and mathematical knowledge, which also is characteristic for and intermediate frame for task design. Note however, that it is important to emphasize that each FP can be solved using different approaches (Albarracín & Gorgorió, 2014) depending on the extra-mathematical knowledge available to students at each educational level (Ärlebäck, 2009). Hence the FPAT framework in addition also exhibited *domain-specific task design characteristics* (cf. Watson & Ohtani, 2015) related to the FP at hand and its context.

The inspiration for the FPAT categorization to include a geometrical representation comes from the work by Anderson & Sherman (2010) with university Economics and Business students. They apply the Fermi method to, and represent an *a priori* analysis of, the task *How many hotdogs are consumed at the Major League Baseball (MLB) games each season in the US?* using a simple graphical diagram (see Albarracín & Ärlebäck (2019) for more details). In the FPAT, we have developed their simple graphical representation and introduced a representation separating the different activities in Table 1 above. To illustrate how our graphical representation enhances the accessibility and directness of the framework, we present a short example analogously to the task discussed by Anderson & Sherman (2010), namely:

TABLE 1. *The FPAT characterization*

Activity	Geometrical representation	Activity Description *Students obtain the quantity by engaging in . . .*
Guesstimation	Ellipse	...a mental process giving a rough solution through guessing based on previous experiences.
Experimentation	Trapezoid	...in-and-out-of-school experimentations and investigations, including making measurements.
Looking for data	Rectangle	... searching for numerical information in external sources.
Polling or statistical data collection	Hexagon	...suitable ways of selecting, collecting and analyzing statistical data.

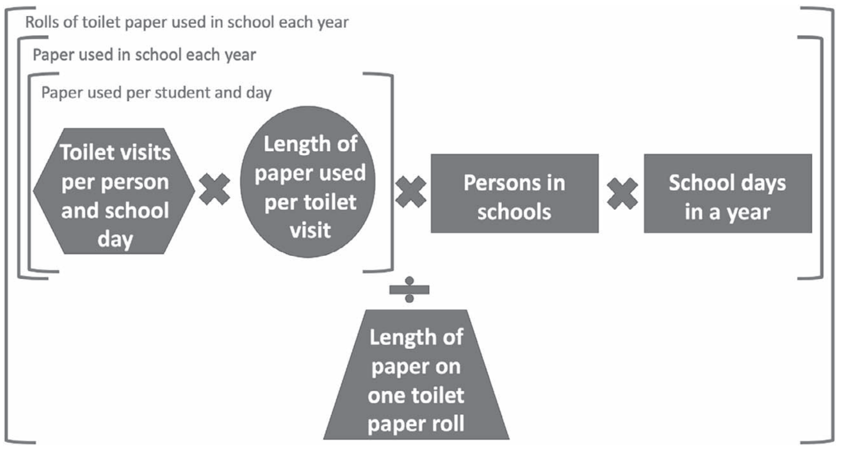

FIG. 2. An *a priori* analysis of The Toilet Roll Paper Task characterized using FPAT.

How many rolls of toilet paper are used in the schools in your country every year? By making an *a priori* analysis of, and explicitly differentiating between the sub-problems the students reasonably have to engage in to solve the problem, Fig. 2 illustrate a FPAT for the so-called *Toilet Roll Paper Task* in terms of the structure of the solving process. In the FPAT the different sub-problems are delimited using square brackets. As illustrated in Fig. 2, the sub-problem to determine the amount of [*Paper used per student and day*] is to be calculated by multiplying the number of *Toilet visits per person and school day* with the *Length of paper used per toilet visit*. The result of this calculation ([*Paper used per student and day*]) is then going to be used as input to the calculation of the sub-problem determining the amount of [*Paper used in school each year*], by multiplying [*Paper used per student and day*] with both the number of *Persons in the school* and the number of *School days in a year*. Finally, the answer to the original question, the number of [*Rolls of toilet paper used in school each year*], is given by dividing the answer to the sub-problem [*Paper used in school each year*] with the *Length of paper one toilet paper roll*.

Importantly, there are different types of activities one can consider engaging in to achieve the numerical values needed in solving the different sub-problems. For example, rather than using a standard way of guesstimating based on previous experiences, consulting official school and governmental records might provide the number of persons in the schools (staff and students) and the number of school days in a year.

To determine how many toilet visits a person make each day, a (anonymous) statistical survey or data collection could be conducted, and due to the potential delicate nature of the amount of toilet paper usage, a guesstimation could be used to estimate the average use of toilet paper per toilet visit. In the case of finding the length of the paper on a toilet paper roll, one can engage in trying to calculate the total length using a geometrical argument or, as suggested in Fig. 2, conduct some investigation of a (number of) physical toilet paper roll(s), either involving unrolling the paper roll and measure the length or some weighing process.

In terms of using the FPAT as a tool for analyzing students' work, Fig. 3 provides an example of an a posteriori analysis of (a group of pre-service primary teacher) students' work on the *Toilet Roll Paper Task*. Figure 3 provides the pre-service primary teacher students' own generated representation of their solution after having solved the FP. The students participated in a 3-h workshop on FPs in which they were introduced to the FPAT framework, and Fig. 3a is their first unaltered attempt in using the framework for representing a solution to a FP. The fact that what is presented below is the students first unaltered attempt explains that there are shortcomings in the FPAT such that (a) not all sub-problems are named and delimited adequately; (b) the use of square-brackets in not consistently applied; and (c) the only activity used to get the needed numerical values to solve the FP is guesstimation. However, the FPAT do capture and convey some of the key components in the students' approach in solving the FP. In Fig. 3b the students' representation of their solution in terms of their constructed FPAT is unpacked and explained. Here, square brackets represents either subproblems or a single quantity identified and delimited by the students as a subproblem to be determined it in its own right. Curly brackets are the authors' identified subproblems the students actually calculate using the specified quantities as represented in their FPAT. The indexing letters A, B and C have been introduced to facilitate referencing the different subproblems and quantities in the solution.

The pre-service primary teacher students choose to answer the FP in terms of calculating the volume of the yearly needed amount of toilet paper. This amount of toilet paper was estimated by multiplying the {*Number of bales of toilet paper rolls used per year*} (which corresponds to the larger square-bracket sub-problem on the left, and which by the student incorrectly was named [*A. Number of toilet paper rolls*] by the students) with the [*D. Volume of one bale of toilet paper rolls*]. To do this:

- First, the {*Number of toilet paper rolls used per year*} is calculated by multiplying the {*Numbers of toilet paper rolls used in school per week*} by the [*B3. Number of weeks students attend school in a year*], were the {*Numbers of toilet paper rolls used in school per week*} is given by *B1. How often toilet paper rolls are replaced each week per toilet* multiplied by *B2. How many toilets there are*.
- Secondly, the {*Number of bales of toilet paper rolls used per year*} is then calculated by dividing the {*Number of toilet paper rolls used per year*} by the [*C. Number of toilet paper rolls on one bale*].

Finally and thirdly, the annually needed amount of toilet paper in terms of volume of toilet paper, is calculated by multiplying the {*Number of bales of toilet paper rolls used per year*} with the [*D. Volume of one bale of toilet paper rolls*].

5. FPAT as facilitator for interdisciplinary collaboration

In identifying and outlining the four activities in the FPAT framework (*guesstimation*, *experimentation*, *looking for data* and *polling or statistical data collection*), we explicitly have drawn on literature from the STEM disciplines. This shows that FPATs as a tool for designing tasks in terms of a *domain-specific frame* (cf. Watson & Ohtani, 2015) can support teachers and researchers in identifying opportunities to connect and bring in other STEM-related contexts, subject matter and content into the teaching

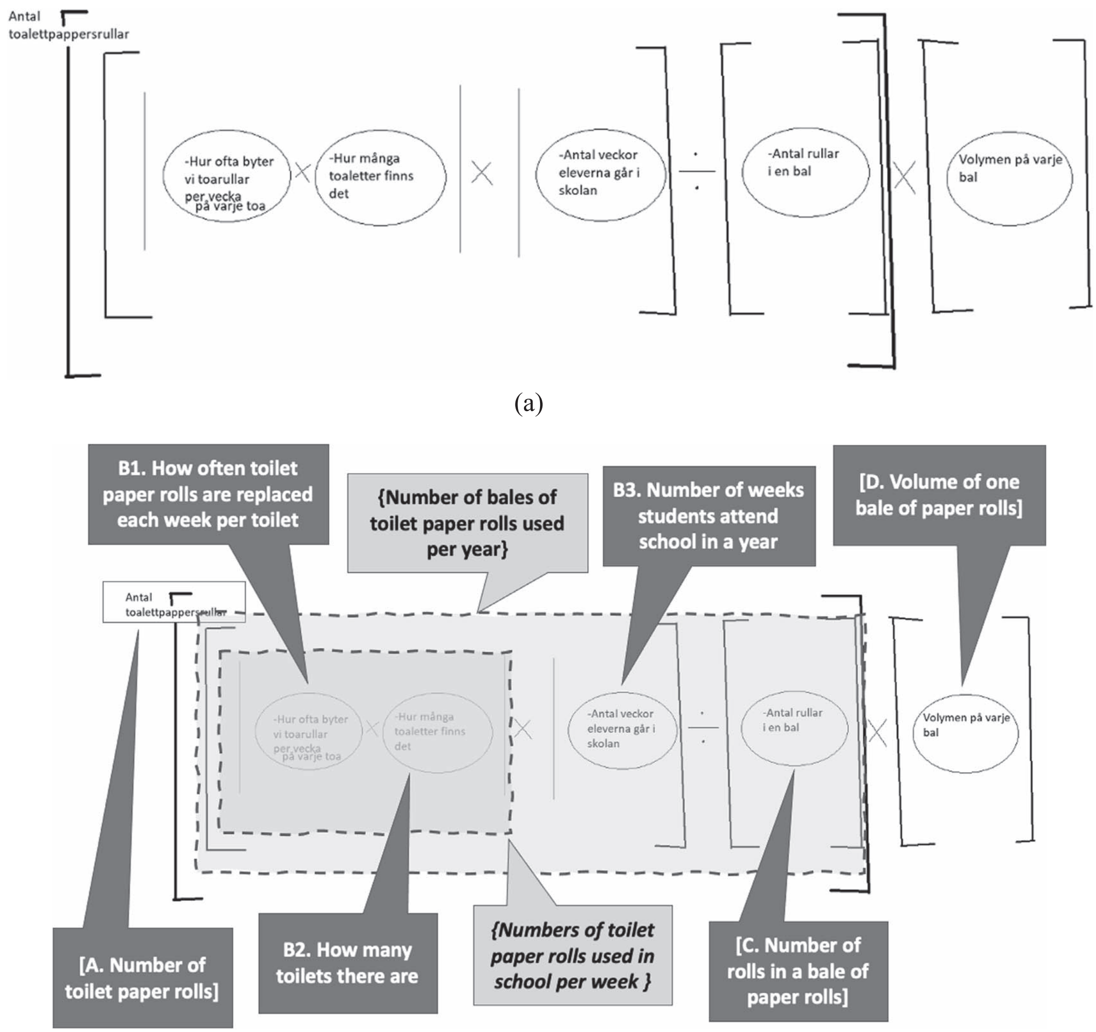

FIG. 3. a. The unaltered a posteriori analysis made by a group of pre-service primary teachers of their solution to the Toilet Roll Paper Task using the FPAT framework for the first time. b. The unpacked and translated *a posteriori* analysis of the pre-service primary teacher group's solution to the Toilet Roll Paper Task using the FPAT framework in panel a.

and learning of mathematics. Hence, FPs can be used as activities in which STEM disciplines are integrated into the teaching and learning of mathematics to increase students' achievement in both the STEM disciplines and mathematics, centered around meaningful real-world challenges and problems (Roehrig *et al.*, 2012). However, as illustrated in the Toilet Roll Task above, also content and topics from the social sciences potentially can come to the fore. Sriraman & Knott (2009) for example stress the suitability of FPs potential to make aspects of governmental and corporate policies visible and available for critical scrutiny. FPs, as miniature-modelling problems (Robinson, 2008), make them accessible tools for investigating the models used in for example economics, public health, history, sociology and political science. Although less experienced problem-solvers may face challenges in grasping and structuring specific given problem contexts, they can successively develop their FP solving competence to overcome this by for example engaging in carefully designed sequences of FPs, to be able to address socially relevant and critical problem situations. Here, the FPAT activates can support and facilitate the direct connections to areas outside traditional mathematics in various ways through the different nature inherent in the four activities. For a more detailed discussion see Ärlebäck & Albarracín (2019).

6. FPAT as a tool supporting teachers to implement FPs

From a teacher perspective, the clear structure and accessibility of the FPAT framework offers a concrete tool for thinking about how to bring modelling into the classroom using problems posed in meaningful and relevant contexts and potentially overcoming some of the difficulties discussed in the literature (Borromeo Ferri & Blum, 2013). A FPAT characterization of a task prepares the teacher and provides input on what to expect, what to prepare, as well as picturing what different problem-solving routes the students might take. In this way the FPAT reduces some of the challenging aspects associated with the openness of modelling problems. In addition, the FPAT graphical representations of the four potentially involved activities (*guesstimation*, *experimentation*, *looking for data* and *polling or statistical data collection*) provide a quick overview of the envisioned and potential development, as well as way to a posteriori analyze and represent students' work. For example, in the context of working with pre-service primary teachers, Fig. 3 provides information on how to support students in developing and strengthen their abilities in terms of (i) what aspects of the problem solving process to emphasize; (ii) what the relevant sub-problems needed to solve the FP are; and (iii) how to think productively about involving other types of activities and making connections to other subjects. Taken together, this suggests that the FPAT potentially and productively can function as an *intermediate frame for task design* (cf. Watson & Ohtani, 2015) and support teachers in making conscious choices in designing, planning, implementing and evaluating both single FPs, as well as sequences of activities involving FPs, focusing on given mathematical contents and curricula (and possible interdisciplinary) goals.

However, supporting teachers in overcoming the difficulties associated with implementing modelling is not trivial (Borromeo Ferri & Blum, 2013). Future research needs to study what challenges arises, and what support or training can facilitate to overcome these, when teachers try to use the FPAT as an *a priori* and a posteriori tool in their classroom practice engaging their students in modelling activities.

7. Concluding remarks

In this paper, we have discussed FPs using the models and mode*l*ling perspective on teaching and learning to elaborate on aspects related to their use as stand-alone activities as well as in sequences of activities. We have also presented the FPAT framework for supporting the design and use of FPs and discussed how this can support and to facilitate (the design of) (i) students' learning within particular mathematics content areas; (ii) students' learning of particular concepts or higher-order thinking skills; (iii) interdisciplinary task (connected to the social sciences and the other STEM subjects); and (iv) for teachers adapting and implementing FPs in their classrooms. With respect to the teaching and learning of mathematical modelling, our initial work has lead us to believe that the FPAT framework has the potential to support teachers and the training of pre-service teachers to overcome some of the difficulties identified in the literature (Borromeo Ferri & Blum, 2013; Blum, 2015). In particular, a FPAT framework provides an approach that allow us to potentially make progress in addressing the challenge of effectively introducing, and maintain a more regular use of, mathematical *modelling* activities in everyday classrooms (Borromeo Ferri, 2021). As such, our discussion and examples have shown how the FPAT framework can be seen to function as a theoretical framework of task design that is applicable at different grain sizes: both at (1) an *intermediate frame size* with its capacity to acknowledge the situational complexity of leaning, and often involve interactions between task, teacher, teaching methods, educational environment and mathematical knowledge; as well as at a (2) *domain-specific frame size* with its capacity to focus on more narrow applicable content in mathematics and other disciplines (Watson & Ohtani, 2015). We are presently engaged in research further exploring these aspects using the FPAT framework as (a) an analytic tool of students' and pre-service teacher's work on FPs; and (b) a tool for pre-service teachers engaged in task design.

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Jonas Bergman Ärlebäck is a mathematics education researcher in the Department of Mathematics at Linköping University, Sweden. His research interests are foremost aspects related to the teaching and learning of, and through, mathematical modelling and the teaching and learning of statistics.

Lluís Albarracín is a professor and Serra-Húnter Fellow in Mathematics Education at the Universitat Autònoma de Barcelona. He is part of GIPEAM (Grup d'Investigació en Pràctica Educativa i Activitat Matemàtica, 2021 SGR 159) research group, and this work was supported by grant PID2021-126707NB-I00 funded by MCIN/AEI/10.13039/501100011033 and by 'ERDF A way of making Europe'. Lluís' current research interests centre on mathematics modelling and estimation processes. He is also interested in the use of videogames in mathematics education.