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REASONABLE AND REASONABLENESS OF ANSWERS: KUWAITI MIDDLE SCHOOL TEACHERS' PERSPECTIVES

ABSTRACT. This study describes mathematics teachers' views on the significance of reasonable answers and how they address this concept in their classrooms. Data were gathered from thirteen eighth-grade Kuwaiti mathematics teachers in two middle schools. All teachers in Kuwait use the national textbook and follow the accompanying instructional plans provided by the Ministry of Education. The results revealed that the overwhelming majority of Kuwaiti teachers reported that an answer needed to be exact to be reasonable. Only three teachers reported discussing reasonable answers and their discussions were in response to student mistakes. None of the teachers addressed the concept proactively, considering reasonableness when planning their lessons or preparing activities or exercises. They defended their lack of attention to this concept by noting that the notion of reasonableness is not included in the Kuwaiti national curriculum.

KEY WORDS: international studies, number sense, reasonable answers, reasonableness, teachers' beliefs of number sense, teacher education

Reasonable and reasonableness are illusive terms. Yet the ability to judge the reasonableness of an answer has long been valued in the mathematics community. It encourages reflections on results and the processes used to obtain them (Sowder, 1992a). It provides a check on computations. It is also used in daily lives, such as reflecting on the reasonableness of the total cost of a bill in a restaurant or grocery store.

Recognizing reasonable answers has been emphasized in many major mathematics documents in the United States of America. Early in the 20th century, the National Committee on Mathematical Requirements included judging the reasonableness of results as one of the practical aims of mathematics education (NCMR, 1923). Fifty years later, the National Council of Supervisors included identifying reasonable answers as one of the 10 basic skills that every person needs to develop (NCSM, 1977, 1989). In 1989, the National Council of Teachers of Mathematics highlighted the need to develop number sense in order to judge the reasonableness of an answer. "Intuition about number relationships helps children make judgments about the reasonableness of computational results and of proposed solutions to numerical problems. Such intuition requires good number sense" (NCTM, 1989, p. 38). More recently, the NCTM *Principles and Standards of School Mathematics* echoed the connection between number sense and

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reasonableness, and stated that all students should be able to "judge the reasonableness of results" (NCTM, 2000, p. 32).

Research related to reasonableness and reasonable answers has been limited. These studies have investigated the reasonableness of estimates or the reasonableness of exact calculations (Bell et al., 1981; Hiebert and Wearne, 1986; Reys et al., 1980; Vance, 1986; Wyatt, 1985; Sowder, 1992a,b). This study reports middle school Kuwaiti teachers' views of the concept of reasonable answers and how they address this concept in their classrooms.

Theoretical discussions of the reasonableness of answers have focused on the meaning of reasonableness and the criteria upon which students judge the reasonableness of answers. An answer is reasonable if it is likely that an answer that has been obtained (or is provided) is a correct response for a mathematics problem. The literature has identified two main criteria used for judging the reasonableness of an answer: (1) *number relationships and the effect of operations* and (2) the *practicality of the answer* (Gagne, 1983; Hiebert and Wearne, 1986; Johnson, 1979; Reys, 1985; Willis, 1992; McIntosh and Sparrow, 2004).

Background for the Study

Number relationships and the effect of operations are critical components of reasonableness (Wyatt, 1985). Thoughtful consideration of the magnitude of numbers and how they are related to each other as well as the effect of operations on numbers (including whole numbers, fractions, and decimals) provides insights about the boundary of a reasonable answer. Students using these components are more likely to identify a range of reasonable answers. For example, understanding a magnitude of decimals and the effect of multiplication helps students recognize that an answer of 29.135 for multiplying 534.6 and 0.545 cannot be reasonable, because multiplying by 0.545 would be around half of 534.6. Reys (1985) emphasized that an unreasonable answer is not limited to a result that is far from the correct answer. For example, 31.94 cannot be a reasonable answer for 1.99 times 15 because 1.99 is close to but less than 2, and 2 times 15 is 30. Therefore, 30 is the upper bound for a reasonable result. This illustrates one of the limitations of using a fixed percent to determine if an answer is reasonable. The previous example, 31.85 is within 10% of the exact answer but is still unreasonable because it exceeds an upper bound of 30.

Practicality of the answer is a second criterion for judging reasonable answers. This criterion focuses on the connection between the answer and the data in the real world. An answer is considered reasonable if it makes sense in the real world. For example, an answer of $5.30 for the cost of one week's groceries for a family of four would not be seen as reasonable. A student who arrives at this result should recognize the unreasonableness of this answer and look for a mistake in his or her calculations. Another example for this criterion is producing an unsuitable number for the context (Dougherty and Crites, 1989; Greenes et al., 1993). For example, 25.5 could be an average for a test but could not represent the number of people at a party or cars in a parking lot.

Little is known about Kuwaiti mathematics teachers' views of the mathematics they teach or the way they teach it. The Third International Mathematics and Science Study (TIMSS) (Beaton et al., 1996) reported a general picture of eighth-grade teachers' perceptions of the nature of mathematics and mathematics teaching. The study found that the eighth grade Kuwaiti students who participated in the TIMSS had mathematics teachers who demonstrated a strong belief in learning mathematics by repetitive practices. Approximately 70% of the teachers believed that memorizing formulas and procedures is important in learning mathematics; less than 50% of the teachers believed that creative thinking and the ability to provide reasons to support conclusions are important.

Although recognizing reasonable answers is valued in many countries, attention to reasonable answers is not a high a priority in many other countries (Reys and Noda, 1994; Yang, 2005). The term reasonable answer does have an equivalent meaning in Arabic, and the Arabic expression is "ejabah ma'aquolah". Yet Kuwait is a country that has given very little attention in the mathematics curriculum to developing reasonableness. The Ministry of Education determines the mathematics curriculum, and one national textbook is used at each grade. A review of the mathematics textbooks across the grades that are used throughout all the schools in Kuwait, and observations of mathematics classrooms suggest that the national textbooks focus on manipulating mathematical symbols and obtaining exact solutions (Alajmi, 2004). Attention to the reasonableness of answers is neglected in both, mathematics textbooks and teaching in the mathematics classes. The notion of reasonable answers did not come in Kuwaiti mathematics textbooks; there were no activities, examples or exercises that asked students to reflect on the reasonability of their answers. Instead, in some topics that deal with numbers and operation the mathematics textbooks encourage students to evaluate the accuracy of their answers by performing another written computation, such as using multiplication to check the results of a division problem. The focus is not on the reasonableness of answers, but on the correctness of exact calculations.

Mathematics teachers in Kuwait follow the national textbook and the plan of the Ministry of Education about what, when and how to teach mathematics. Therefore, there is little variation among the teachers regarding the content taught in their classrooms. They focus on goals specified by the Ministry. Teachers in middle school and secondary school teach only one subject and they teach it for all the grades in the grade band. For instance middle school mathematics teachers teach mathematics for all the middle school grades 6, 7, 8 and 9. Usually, teachers teach two grades in each year and within four years they get the chance to teach all these grades. For example, a teacher can teach grades 6 and 8, then the next year he or she can teach grades 6 and 7, grades 7 and 8, or sometimes, keep the same grades for two years in a row. Furthermore, teachers work closely together in planning and carrying out their lessons. Teachers in each school share the same workroom, so they have an opportunity to discuss and share ideas from their classes. Teachers have weekly meetings with their colleagues to discuss what they will teach and how they will teach it in each grade. The senior teacher in each school guides these meetings. She or he helps direct the group to reach the goals that are set by the Ministry of Education. During these meetings, the teachers identify general lesson plans for each mathematics topic they teach: they specify the goals for each lesson and identify exercises, homework, and assignment plans. The senior teacher visits the teachers in their classrooms and provides them with feedback to achieve their goals.

There is only one teacher preparation program for middle and high school teachers available in Kuwait, and it is offered by Kuwait University. After teachers graduate from this program, they choose to teach either in middle school or high school. During their undergraduate years teachers acquire around 60 credits in mathematics and 45 credits in professional education requirements. The mathematics classes taken include Calculus I, II, and III, Linear Algebra, Principles of Probability and Statistics, Differential Equations, Abstract algebra (1), Geometry (1), Numerical analysis, and Introduction to topology. Teachers also take two special courses in teaching mathematics for middle and high school under professional education requirements. This single program limits the variation in Kuwaiti mathematics teachers' background. Due to the shortage of mathematics teachers, the Kuwaiti Ministry of Education employs teachers from other countries such as Egypt, and sometimes fills the shortage with elementary mathematics teachers. Teachers who come from other countries usually graduate from mathematics education programs that are similar to what Kuwait University offers.

This study reports Kuwaiti mathematics teachers' views of reasonable answers. More specifically the study addressed the following central question:

What are Kuwaiti middle school mathematics teachers' views of the meaning of reasonable answers and how is this concept addressed within their mathematics classes?

METHODOLOGY

Participants

Two middle schools were selected randomly from the Al-Ahmadi district (one for boys and one for girls). All the mathematics teachers in these two schools were invited to participate in the interview. Seven of the eight mathematics teachers from the boys' school and all six mathematics teachers from the girls' schools participated in an interview. The sample includes the senior teacher in each school. The teachers' years of experience in teaching mathematics ranged from 1 1/2 to 29 years. Additionally, the teachers were of two nationalities, Kuwaiti and Egyptian. Three of the male teachers were Egyptian; the remainders were Kuwaiti (See Table I).

Instrument

The teacher interview was designed to gather information regarding middle school mathematics teachers' views about the reasonableness of answers

TABLE I
The Teachers by gender, nationality, and years of teaching experience

			Years of teaching experience	
Teachers	Gender	Nationality	Elementary school	Middle school
Teacher 1	M	Kuwaiti	0	21
Teacher 2	M	Egyptian	0	29[a]
Teacher 3	M	Kuwaiti	5	7
Teacher 4	M	Kuwaiti	0	10
Teacher 5	M	Egyptian	0	12[b]
Teacher 6	M	Egyptian	0	8[c]
Teacher 7	M	Kuwaiti	6	1
Teacher 8	F	Kuwaiti	0	11
Teacher 9	F	Kuwaiti	0	10
Teacher 10	F	Kuwaiti	12	1
Teacher 11	F	Kuwaiti	0	6
Teacher 12	F	Kuwaiti	0	3
Teacher 13	F	Kuwaiti	0	1 1/2

[a]29 years: 12 in Kuwait, 17 in Egypt.
[b]12 years: 8 in Kuwait.
[c]8 years: 4 in Kuwait, 4 in Egypt.

and how they address this concept in their classrooms. To examine the teachers' views, the interview questions focused on (a) the meaning of reasonable answers, (b) the ways teachers determine the reasonableness of answers, and (c) how they valued this concept. Additionally, the researchers asked the teachers to reflect on their students' responses for some items on a Reasonable Answer Test (RAT) to learn more about these teachers' thought process of a reasonable answer. A complete copy of the RAT along with students' responses is available in Alajmi (2004). To examine teachers' classroom practices, questions focused on how they consider the concept of reasonable answers in their lesson plans, and classroom discussions and other activities, including student assessment.

Data analysis

The analysis of the interview data was conducted in two stages. First, during data collection, after each interview the researcher recorded field notes that reported what she learned about each teacher's views of reasonable answers and how they addressed the concept in their classroom. Second, after collecting the data, the researcher transcribed the interviews and examined those transcripts for emerging themes.

RESULTS

Teachers' views of reasonable answers

The teachers responded to tasks designed to learn their views of the concept of reasonable answers. They were asked to describe what the concept of a reasonable answer meant to them, and provide examples of situations in which they thought an answer was reasonable. Additionally, they explained strategies they used for determining the reasonableness of an answer.

Teacher definitions of a reasonable answer

One teacher provided examples that included one of the criteria for judging a reasonable answer that was used in this study, which was the practicality of the data. She said "a distance cannot be negative; when a student gets a negative answer for a distance, she should realize she made a mistake," and she went on to say "an angle of 70 degrees should be an acute angle; it cannot be an obtuse or right angle." The remaining 12 teachers defined reasonable answers in terms of how they would assess a student answer.

Six of these teachers insisted that in order to be reasonable the answer must be exactly right; the other group gave some allowance for procedural and calculation mistakes. The first group teachers (4 male, 2 female) focused entirely on the exact answer. They insisted that the answer for a mathematics problem is either right or wrong. These teachers argued, "A mathematics problem cannot have two answers. There is one and only one right answer. If it is the right answer, it will be reasonable." Three of these teachers further argued that the correct answer should include all the steps. One said, "If all the procedural steps are illustrated, and the answer is right, the answer is reasonable. " Another said, "The student should show all the procedures and all the steps to get the right answer. In math, the answer is [either] right or wrong."

The teachers provided examples for what they considered a reasonable answer. One teacher gave this illustration: "If we said $5 - 2 = 3$, 4 will not be a reasonable answer. Any answer less than 3 will not be reasonable. The reasonable answer is just 3." While this seems like a strange place to apply the notion of reasonable answer, particularly given the basic fact example by an 8th grade teacher, it does reflect the vision of reasonable expressed by this Kuwaiti teacher. Another said, "If a student is solving a multiplication problem, he will follow specific steps to get the exact answer. " Another explained, If you have a group, let's say X has three elements 1, 2, and 3. So, $1 \in X$, and $4 \notin X$. There are no two answers. It is right or wrong." All these examples emphasized one exact right answer and followed specific procedures.

The remaining group of six teachers believed that an answer could be considered reasonable if it included most of the procedural steps; they left a margin for error for the answer. These teachers focused on the completeness of the procedure. For example, one of them said, "There could be a mistake in the procedure and the final result, but the answer should not be too far from the right answer."

Two teachers provided a rubric for the margin of error for a reasonable answer that was based on the number of procedural steps used in the solution process. One teacher thought it should be 80% correct. She said, "The answer is reasonable if it satisfies 80% of the steps or there is a minor error in the final answer." Another teacher gave more room for error; he thought the answer should be considered reasonable if it is 50% right. He said, "For me, I think of the answer in terms of the students' thinking level. They will not be thinking like me as a teacher. I believe that the students' answers always have something right. It is satisfying for me if the students' answers have 50% of the right answer." He gave this example: "If a student was adding two fractions, $3/5 + 1/3$, if he got the common factor of 15 and did something like $3/15 + 1/15 = 4/15$, for me the answer is reasonable because he worked correctly half of the problem." In this example, the teacher considered the problem "half right" because the student got the concept of the common denominator, even though the answer is far from reasonable.

Another teacher gave this illustration, "if a student was solving a multiplication problem, the answer would be reasonable if she made a mistake in the addition that led to a wrong number in the tens digit, or made a mistake in multiplying two numbers and that caused a mistake in the final answer."

All of these teachers reflected a different understanding of the concept of reasonable answers than what is reflected in the literature. They discussed what they would accept in grading the answer rather than what criteria students could use in judging whether an answer is reasonable. Working half of the problem correctly does not always yield an answer that is reasonable. For example, in the problem of "$3/5 + 1/3$", an answer of 4/15 would not be reasonable because 4/15 is less than 1/3; the reasonable answer would need to be greater than either of the two terms, 3/5 or 1/3. Likewise, a mistake in a mathematics computation may produce an answer that is far from reasonable. For example, if a student working a multiplication problem with two and three digit factors made a mistake in adding the hundreds or thousands digit, he might produce an answer much larger than a reasonable answer. These teachers stressed the completeness of the procedural steps of the solution and allowed a wide range for mistakes in the final results.

Two teachers said that they focused simply on how close the answer was to the right answer. However, the examples they provided suggest that they made allowances for students' common mistakes. One argued that an answer would be considered reasonable "if it is not right, but it is close to the right answer, either up or down." She gave this illustration of her thinking, "$7 + 2 = 9$ is the right answer, but if a student wrote 5, it is reasonable because she did subtraction instead of addition. I would consider the mistake that the student made when he subtracted." The other teacher provided the following example, "6 – (-7). If the answer is -1, it would be reasonable, even if the right answer is 13." Once again some may argue that these illustrations are inappropriate, and not proper representations of reasonableness. These examples are clearly different than the notions of reasonableness reflected in the literature (Johnson, 1979; Yang et al., 2004). Nevertheless, we are reporting actual explanations that Kuwaiti teachers have provided for their classroom illustrations of reasonable and unreasonable answers.

Teachers' [self-reported] strategies for determining the reasonableness of an answer

Teachers most often used an exact answer to judge answers for reasonableness. The teachers also examined the rules and procedures used to arrive at the answer. Their views of the "reasonable answer" became even clearer after the teachers explained the ways they determined the reasonableness of an answer. The majority of the teachers (9) relied upon solving the problem and finding the "right" answer as a standard for judging if the answer was reasonable. In this case, the only reasonable answer in the teachers' view was the exact answer. One said he would use the calculator to examine the answer. Another said "I used scientific procedures, so my answer will be right," which shows this teacher's trust in procedures and rules.

Three teachers said that they would do another computation to check the accuracy of an answer. For example, one said, "if I got a solution of 3 for an equation, I would plug the 3 into the equation to check my answer." Another teacher explained, "If I have division, I would do multiplication."

Teachers' classroom practices regarding the reasonable answer

Given that most of the teachers viewed a reasonable answer in terms of comparing students' responses with the exact answer, and given that they, themselves, primarily followed mathematical procedures to determine the right answer as a means of judging the reasonable answer, it is not surprising to find that these teachers did not do much in their classrooms to foster learning about how to judge the reasonableness of answers. Three of the teachers (one male, two female) said they sometimes came across this concept. The rest of the teachers (10) admitted that they did not address the concept of judging the reasonableness of an answer in teaching mathematics.

The three teachers who sometimes consider the concept of reasonable answers in teaching mathematics appeared to address this concept in response to students' mistakes, rather than proactively. The explanations they gave aligned with the meaning they have for reasonable answers. Two of these teachers reported that reasonable answers have most of the procedural steps, so they focus on mathematics rules and the opposite procedure to help their students examine the reasonableness of an answer. One of them said "Given the problem $-14/-2$. If a student got -7, I would discuss that with him and ask the student to rethink his answer," or ask students to do the opposite operation to check their answers such as multiplication to check an answer for division and addition for subtraction. In this approach these teachers check the correctness of exact calculation when the students produce a wrong answer.

The third teacher was the one who considered the practicality of the answer to be a criterion for judging answers for reasonableness. She stressed that she asked her students to reflect on the practicality of their answers "For instance, if a student got a negative answer for measuring a weight, I would tell her to think about the reasonableness of her answer." This discussion limits students' mistakes.

Teachers agree that judging the reasonable answers is not something they consider in developing their lesson plans, class activities, or problems. The teachers gave two reasons for neglecting this process. First, the concept of reasonable answers is a goal not listed in the national curriculum. Four other teachers argued that they must focus on the goals they need to achieve. One of these teachers said, "I just focus on the goals I have; I actually focus on the concepts I am going to teach." Another one said, "We have our plan and we follow this plan. The reasonable answer is not in this plan." The second reason, they focus on exact answers is because it is the important goal for mathematics and introducing anything about reasonable answers would be confusing for their students. Two teachers explained that they focused on exact answers. One teacher argued, "I focus only on the exact right answer, so students can understand the right answer. It will not benefit them to introduce or talk about reasonable answers."

Two other teachers argued that the concept of a reasonable answer is not important because exact answers are the main goal for teaching mathematics in school. This thinking is consistent with the meaning the teachers seem to hold for reasonable answer, "reasonable answer is the right answer". They insisted that teachers need to concentrate on teaching students how to find the right answer.

> We have the exact answer. It is hard to say that an answer is reasonable or unreasonable. We will reach one answer. If it is right, it is reasonable. If it is not, then it is unreasonable it might be important in the beginning of some math lessons, such as addition. For example, we can say $54 + 22 = 70$. 70 is reasonable but the right answer is 76. But we have 76 is the final right answer and that is what we should focus on.

Surprisingly these two teachers, who saw reasonable answers as not important for students in school, acknowledge its importance for students' lives. One of them said, "Estimation would help students in buying their stuff from the supermarket. They can estimate the total prices and check if they have enough money or not." Another one, however, argued that teachers need to concentrate on the right answer. He acknowledged, however, that a reasonable answer would be useful for students in their everyday lives. He explained this conflict in his opinion, "rounding would be helpful for students in real life, buying and checking prices. [But] learning in school is acquisition of knowledge." These two teachers have developed a view of reasonable answers in school which is the exact answer and what they focus on in their classrooms, but outside the classrooms reasonable answer and estimation can be important. They distinguish between what their students need in the school and what might be useful for them in their life. In other words, school mathematics for them requires *right* answers; *reasonable* answers are of value only in everyday life outside school.

Eight teachers suggested that they might encourage their students to examine their answers for correctness by doing another computation. Strategies these teachers used to encourage their students to examine the correctness of their answer include:

1. Do the opposite operation, such as multiplication to check an answer for division and addition for subtraction.
2. Substitute the answer in the equation to check if it is right or not.
3. Do the operation in two different ways. For example, "in comparing two fractions, if we compare by finding the common denominator, we can compare the same fraction using cross-multiplying to check whether the answer we got in the first way was reasonable."

Additionally, teachers emphasized mathematics rules to help the students to judge the reasonableness of an answer. For example, one rule teaches students that the sum of two negative numbers should be negative.

Only one teacher encouraged her students to judge an answer based on the practicality of the data. For example, she reported that a student cannot get a negative answer for a distance or a weight.

Since teachers did not focus on the concept of a reasonable answer, none of the teachers asked the students to explain the reasonableness of their answers. Nor did they assess their students on this issue. All the teachers concurred that their students would explain the procedures of getting the right answers orally. One of the teachers argued, "Students will explain the steps they learned in class to get the answer." The teachers said, the students would be more likely to provide oral explanations than written ones because "we rarely ask for written explanations. The students cannot express themselves very well. Additionally, they have difficulty in language."

These teachers did not assess their students' ability to recognize the reasonableness of an answer. Two argued that they have other important goals and there is no time to spend on assessing students on this concept. Their curriculum is driven by the textbook and the goals established by the Ministry. "We do not have time for it. I have a lot to do, and I need to finish all the material in the textbook."

Teachers' reflections on students' explanations for judging the reasonableness of answers on some items on the RAT

To get inside the teachers thinking of reasonable answers the researchers asked them to reflect on eight grade students' responses on a Reasonable Answer Test (RAT). On the RAT eight grade students were asked either to judge an answer for reasonableness or to select the reasonable response from a set of responses in less than a minute. The researchers provided the items and a summary of the students' responses and asked the teachers to reflect on these responses. The summary of the students' responses included the students' common explanations that students provided for the each item.

Item 1: The P.E. teacher has a 210 KD budget to prepare for the school's annual party. He spent 4/5 of the money; he asked the management team to find **out** how much money he spent. The managing team did some calculations, then decided that 262 KD was spent. Is their answer reasonable? Why?

Student response: students who answered this item saw that the answer was not reasonable because the school cannot spend more than the budget.

All of the teachers agreed that the *students' responses* were reasonable. However, none of them reflected on the fact that students could judge the reasonableness of the answer provided based on their understanding of the numbers and the operation involved.

Three described the students' response as an excellent way to judge the answer. Three others noted that it was better than working the problem. They all emphasized the speed in using this strategy. "What they did is better than solving the multiplication. It is a fast way." One of the teachers praised the students' ability "to solve this problem in less than a minute," given that they were "not used to this type of question." One teacher appeared to trust working the problem over judging the reasonableness of the answer provided, although she acknowledged that the students' strategy was faster. "For a fast way, their response is reasonable, but I am not sure if the reasonable answer is better than the worked solution."

Item 2: Dalal used a calculator to solve some mathematics problems; she got the right numbers. However, she forgot to place the decimal point in the results. Your task is to place the decimal point so the answer is reasonable.

$534.6 \times 0.545 = 291357$

Student response: almost all of the students placed the decimal point 4 digits from the right. Their explanation was based on the counting rule (the decimal point is after one digit on the first number and after 3 digits on the second number, so it will be after 4 digits on the result).

All of the teachers argued that the counting rule is the only way to place the decimal point and that is what they taught the students. None of them talked about ways of judging the reasonable magnitude of the answer as a way of placing the decimal point. For example, multiplying 534 by about one-half, and concluding that the answer must be between 200 and 300, so the only reasonable answer is 291.357. Instead the teachers reiterated a rule approach, one said "That is what we teach them. There is not a different way to do it. That is what is in the book." Another concurred, "This is the way. It is the counting rule for placing the decimal in the answer."

Three teachers argued that students should provide an explanation for the general rule and provided this example "$10 \times 1000 = 10{,}000$, and that is why the decimal point will be after four digits". The teacher here used $10 \times 1000 = 10{,}000$ as a short way for $1/10 \times 1/1000 = 1/10{,}000$. One of these teachers explained, "We explain to the students in the first period of multiplying two decimal numbers why we place the decimal after one or two digits ($10 \times 1000 = 10{,}000$), but later we just focus on applying the rule."

Seven of the teachers realized that the rule did not work in this item because of the missing zero; they relied upon working, or at least partially working the problem to see that the final zero was missing. Three of these teachers thought that some students should note the missing zero. One of them said, "I think some students should see that $6 \times 5 = 30$ and that means there is a missing zero. So they will place the decimal after 3 digits instead of 4." Another teacher argued that the only way to place the decimal correctly was by working the problem. "The only way to do this problem is by working it. Students need to multiply, and then place the decimal, so they can get it right." He further elaborated, "What the students did is a mechanical, ineffective way to place the decimal; they need to work it in order to see how many zeros are missing."

Two teachers thought the problem was too difficult for this age level and argued that most of the students would not be able to place the decimal correctly. Two other teachers said that they would not deal with this case in their classrooms because it is too difficult. "We do not deal with these kinds of questions; we focus on simple items on the test. When I think the question will have a high level of math, I will not bring it up in the test or in the classroom."

Item 3: Choose from the numbers in the box a number that makes the statement right for you. Then explain why you chose this number.

Number Box

$3\frac{8}{9} \div \_ > 4$

$3\frac{3}{4}$	1.54	$1\frac{1}{5}$	$\frac{3}{5}$
0.9	0.05	2.5	

Student response: The most frequent choice for this item was $3\ \frac{3}{4}$. Students failed to provide an explanation for their thinking. Most of them simply said, "The answer is reasonable," or "it is the right answer."

Twelve teachers reflected on the students' responses for this item. Ten of the teachers agreed that the item was difficult because it dealt with dividing a mixed number composed of a whole number and a fraction. One of the teachers argued "It is hard; it includes two operations. First, changing the division to multiplication and then doing all the steps. They cannot do it." Another one argued "It is hard. You want them to divide a fraction, then the answer should be bigger than 4." One of the teachers did not understand the item. She said "How can the answer be bigger than 4 when you divide?"

Six teachers suggested that the only way to approach this problem would be to try all of the choices. One of them explained, "If I would think about it, I would try all of the choices." Another teacher said, "I myself would need to work it out. I would need to try all of these choices."

Four teachers reflected on the students' choice of 3 3/4 as the reasonable answer. Two of them suggested that the students chose 3 3/4 arbitrarily. "It is a hard question. I think they just made a random choice." The other two teachers argued that the students based their choice on the form of the answer. One of them explained, "When the students failed to solve a problem, they will look for the similarity of the numbers. 3 3/4 and 3 8/9 both have the same form, whole numbers and fractions."

Two teachers thought the item was not appropriate for these students. Another one argued, "You just gave them less than a minute to solve the problem. You are looking for very smart students who can try any of these choices fast." He further argued, "This is not required for the students to solve the problem in less than a minute. We do not deal with it." The other teacher said, "We are not dealing with these high level complicated ideas in our classrooms. We do not have time to do it. Maybe at the end of the year, if I have time, I may discuss complicated mathematics ideas." Note that both seemed to approach this as a problem that needed to be *solved* rather than one that simply asked for a reasonable answer.

Teachers' reflections on students' responses provided more evidence of their view of reasonable answers. They emphasized standard procedures and rules for approaching the problems rather than reflecting on the reasonableness of the answer. Their responses indicate procedural thinking and limits development for number sense.

CONCLUSIONS

Participants in this research were teaching in the traditional educational system in Kuwait, where the focus is on following standard algorithms and finding exact answers. Additionally, they followed the national textbook and the plan that is provided by the Ministry of Education, which does not include reasonable answers as part of the curriculum. Their limited exposure to the concept of reasonableness and reasonable answers influenced their thinking about this concept. Although teachers were consistent in applying their notion of reasonableness, it was evident that the overwhelming majority of Kuwaiti teachers had a concept of reasonableness that was dramatically different from the literature. The teachers stressed procedural rules and doing the calculation to solve the problem as the ways to check the reasonableness of an answer. This rationale to support their position became clear in the examples they provided and the strategies they used.

The Kuwaiti middle school mathematics teachers' common view of a reasonable answer as an exact answer and their focus on following mathematical procedures is aligned with the finding of TIMSS (Beaton et al., 1996) about Kuwaiti mathematics teachers' beliefs about learning mathematics. Kuwaiti teachers also viewed school learning of mathematics as separate from real-life mathematics. Several teachers acknowledged the value of determining reasonable answers without computation in real-life, but did not see this as a goal of mathematics learning in school. Their view reflects situated learning, and is similar to the 'school versus street' learning (Lave, 1988; Carraher et al., 1985, 1987).

The teachers that viewed the result of summing the numerators and denominators of two fractions to produce a reasonable answer raises questions about the depth of mathematical knowledge of these teachers. Their willingness to associate a percent of correct procedures with reasonable answers is at the least worrisome. Such responses raised questions about the number sense of these teachers and their overall understanding of mathematics. No prior research was found about Kuwaiti teachers' knowledge of mathematics content. Research in other countries has documented that mathematics teachers may apply and manipulate mathematics rules but may not be able to justify or explain why the mathematics procedures worked (Ball, 1990; Ma, 1999; Simon, 1993).

Given the teachers' focus on finding exact answers, it was not surprising to find that only three teachers used the reasonableness of answers in responses of their students mistake. These teachers mostly focused on doing another procedure or memorized rule to help their students reflect on the correctness of their students answers. These results confirm Johnson's observation that teachers usually discussed reasonableness after an error occurred and if students get the correct answer reasonableness will not be discussed (1979).

RECOMMENDATIONS

This is the first research study to address the concept of reasonableness in Kuwait. Consequently, these results need further investigation for Kuwait mathematics teachers' views of reasonableness, reasonable answers and number sense. Although it is a single study, this research suggests the view of mathematics teachers in Kuwait on reasonable answers is significantly different from the view described in the literature.

The mathematics Kuwaiti teachers' views of reasonable answers are a reflection of the education system and the general cultural view for mathematics, where they were raised and trained. The Kuwaiti education system focuses on procedural knowledge, by emphasizing specific rules and algorithmic steps to follow calculating the exact answer and drilling students to help them master mathematics skills. Generally, people in Kuwait think of mathematics as rules that need to be memorized to get the exact answer. This heavy emphasis on procedural rules impacted current Kuwaiti teachers' understanding of what constitutes a "reasonable" answer.

There is a need to help Kuwaiti teachers change from focusing on an exact answer and the number of steps in a procedure to reflect more and use their understanding of number relationships, and the effect of operations and the practicality of the answer to examine the answer for reasonableness without doing another procedural computation. This can be done through pre-service teacher education. Teachers should be involved in workshops that focus on helping them to expand their views of reasonable answers and provide activities that help teachers learn how to judge the reasonableness of an answer and reflect these ideas in their teaching.

To support the change in Kuwaiti teachers' views of reasonable answers, the concepts of reasonableness and estimation needs to become visible in Kuwaiti national mathematics textbooks. The emphasis on the national textbook needs to be shifted from focusing on standard algorithms to developing number sense and providing students and teachers with the opportunity to think of alternative methods to determine reasonable answers. Hopefully this study will help stimulate some changes in future mathematics textbooks used in Kuwaiti schools.

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