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ORIGINAL ARTICLE

Exploring the negotiation processes when developing a mathematical model to solve a Fermi problem in groups

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Abstract
Previous research has highlighted the importance of social relationships in mathematical group work while working on modelling activities. This study analyses the interaction of sixth-grade students in Primary Education (11 to 12 years old) carrying out a modelling task in groups with a Fermi problem used as the modelling activity. The focus of the study was to explore how students develop a mathematical model to solve a Fermi problem in groups. The data collected mainly came from the group discussions, although the students' productions were also considered. The results show that a variety of factors can influence group work and that model development is based on one student introducing an initial model and then, through social interaction with the other group members, the model is improved to develop a solid strategy that may be useful for solving the problem at hand.

Keywords Fermi problems · Mathematical modelling · Group work · Interaction

Introduction

Since the publication of Pollak's seminal work (Pollak, 1979), mathematical modelling has been an object of interest in mathematics education, as a way of introducing activities that highlight the deep relationship between mathematics and the world around us. Given that mathematical modelling activities serve as a didactic vehicle both for developing modelling competency and for enhancing students' conceptual learning of mathematics (Blomhøj & Kjeldsen, 2013), there is an increasing interest in introducing new activities that include mathematical modelling in the

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curriculum at different educational levels (Vorhölter et al., 2014). Furthermore, advances in research on mathematical modelling have led to its learning being seen as a necessary practice for learning mathematics, which prepares students for their everyday and professional lives (Hernández-Martínez & Vos, 2018).

Linking the domains of reality and mathematics and navigating between them is the main characteristic of the modelling process. This gives students the opportunity to connect their knowledge with the problem set and challenge themselves to find strategies that adapt to the situation under study. To this end, the most widespread form of work organisation in the classroom is group work, to the extent that it is the recommended option for organising modelling activities (Geiger et al., 2022). Modelling involves social interaction between students. During model development, problem solvers interact and communicate with others, and experience a series of assimilations, which are ways of reaching agreements about the conceptualisation of the model (Sevinç, 2021). However, the literature also points out that it is important to pay attention to learners' individual characteristics, as these affect the decisions made during group work when it comes to the generation of models (Kaiser & Maaβ, 2007).

In this research, we started with the assumption that each of the students who were part of the working group would have their own vision of the model they were developing and each of the individuals would make his or her contributions by interacting with the real phenomenon under study and through interaction with co-workers. We focused our interest on mathematical modelling activities where primary school students generate mathematical models while working in groups. Specifically, in this article, we present research carried out to explore the ways in which primary school students negotiate the process when solving a Fermi problem as a modelling activity.

Characterised as open-ended questions, Fermi problems offer little or no specific information to guide the problem-solving process (Efthimiou & Llewellyn, 2007), but rather emphasize the need to think carefully and analyse the problem situation at hand. Previous research has shown that not all students are capable of generating individual plans to solve the problem presented, but that they can solve it through group work and communication with others (Albarracín & Gorgorió, 2018). Moreover, it has been observed that primary school students working in groups solve Fermi problems by creating new mathematical concepts and generating their own mathematical models (Albarracín & Gorgorió, 2019; Peter-Koop, 2009). From the theoretical perspective, mathematical models developed by students are complex conceptual systems (Lesh & Harel, 2003). The purpose of this research was to determine the nature of students' discussions, negotiations and contributions during the generation of mathematical models while working in groups. This study focused on group discussion around the initial formulation of a mathematical model while solving a Fermi problem by working in small groups. Thus, the research goal of this study was as follows:

- To characterise the exchange of leading ideas involved in the negotiation processes leading to the development of a mathematical model during the interaction of students doing group work while solving a Fermi problem.

Background literature

This section is divided into three main parts: mathematical modelling, Fermi problems, and group work and interaction.

Mathematical modelling

To promote the learning of mathematics in the early stages, we consider that it is essential to present students with a wide range of mathematical content, from more specific to more abstract concepts and using realistic contexts as a vehicle for their development. During the mathematics learning process, students must have the opportunity to develop and apply mathematical concepts and procedures in situations close to their own reality so that they understand the abstract concepts that describe them (Gravemeijer, 1994). Thus, we chose mathematical modelling activities to promote students' conceptual learning of mathematics (Blomhøj & Kjeldsen, 2013).

Research on mathematical modelling has become very diversified in terms of both goals and approaches (Abassian et al., 2020). There are studies where the characteristics and features of final models created for specific activities are examined, but those studies do not discuss the related modelling process (Shahbari & Daher, 2016; Yanagimoto & Yoshimura, 2013). Other studies focus on how students create mathematical models to solve a problem, with differing viewpoints (Borromeo Ferri, 2006; Galbraith et al., 2005; Hankeln, 2020). In mathematics education, it is accepted that modelling processes have a cyclical nature (Blum & Leiss, 2006; Doerr & English, 2003; Galbraith & Stillman, 2006; Kaiser & Stender, 2013; Maaß, 2006). When students are trying to solve a modelling activity, they go through different stages that transfer content between two domains (reality and mathematics) by moving through what is known as the modelling cycle (Blum & Leiss, 2006). However, it is well known that the process of solving a mathematical modelling problem does not rigorously follow the modelling cycle, and that the cycle is only a theoretical reference that does not show most of the actual work done by the students (Cai et al., 2014).

Considering the complexity of the mathematical modelling process, we were interested in finding out how students construct a mathematical model from the ideas contributed by each student when working in a group. We followed the definition established by Lesh and Harel (2003), where mathematical models are considered to be conceptual systems expressed using a variety of interacting representational media, with the purpose of constructing, describing or explaining other systems. Models include a conceptual system for describing or explaining the relevant mathematical objects, relations, actions, patterns and regularities, and the accompanying procedures for generating useful constructions, manipulations and predictions. Considering the Lesh and Harel (2003) definition, we understand a model as a way of representing a reality through the use of various forms of expression (words, drawings, schemes…) and consisting of two types of elements: conceptual and procedural. Its purpose is to describe a system—usually a complex one—that students need to understand. In the case of problem solving, the model helps students to represent the situation that arises and to find a reasonable solution based on mathematical processes. At primary school level, students do not possess mathematical language tools such as algebra or graphical representations of functions to express their models. However, they are able to use natural spoken or written language, drawings and calculations to illustrate their ideas while solving Fermi problems (Ferrando et al., 2017), thereby making Lesh and Harel's definition (2003) the aptest way of displaying students' productions during a modelling activity.

Fermi problems

According to Ärlebäck (2009), Fermi problems are 'open, non-standard problems that require students to make assumptions about the problem situation and estimate relevant quantities before carrying out often simple calculations' (p. 331). On the other hand, Efthimou and Llewellyn (2007) characterised Fermi problems by the way they are formulated, as they are always posed as open-ended questions that offer little or no information to the solver.

This type of question was originally used by the physicist Enrico Fermi to demonstrate the power of deductive thinking, as well as to prepare students for experimental work in the laboratory. Having been used as a didactic tool in class, they transitioned to different usages in other disciplines (Ärlebäck & Albarracín, 2019). The procedure proposed by Fermi was to decompose the original problem into simpler subproblems and to reach a solution to the original question by making reasonable estimates or educated guesses after first considering the individual subproblems (Carlson, 1997). In the literature, this way of working is known as the Fermi (estimates) method. Thus, Fermi problems are those questions that ask students to estimate quantities in the real world by following the Fermi method. In any case, the usual way of employing these problems in the mathematics classroom promotes the use of different types of activities to generate the estimations that respond to each sub-problem. (Albarracín & Ärlebäck, 2019) have identified four activities, these being guesstimation, data search in information sources, data collection and statistical processing, and measurement. This specific way of using Fermi problems in the mathematics classroom makes it possible to relate their solution to mathematical modelling activities, as shown by several studies (Ärlebäck, 2009; Czocher, 2018; Albarracín & Gorgorió, 2014; Ferrando & Segura, 2020; Peter-Koop, 2009).

The Fermi problems introduced in mathematics classrooms are based on a real context that students can relate to, such as the number of cars stuck in a traffic jam on a motorway (Peter-Koop, 2009). Fermi problems promote the process of modelling a phenomenon while students learn to make estimations (Albarracín & Gorgorió, 2013; Robinson, 2008). Peter-Koop (2009) used Fermi problems with primary school students (aged 10 to 12) to analyse their solving strategies. Based on her research, she concluded that students solved problems in several different ways, that they developed new mathematical knowledge to arrive at their solutions, and, finally, that the solving processes used by students had a multicyclic nature and followed the modelling cycle.

Albarracín & Gorgorió (2019) worked on the construction of the models used by primary school students (11 to 12 years old) to solve Fermi problems. In this research, they observed that there were various strategies and ways of building models to solve a Fermi problem. In Ferrando and Albarracín (2021), the models generated by students of different ages to estimate the number of objects that can be placed on a plane were compared. These authors observed that the models generated by upper-level primary students (10-year-olds) were based on a static view of how each object occupies its portion of the surface (grid distribution or use of a reference point). Henze and Fritzlar (2009) studied the working processes involved when solving a Fermi problem in groups. They maintain that Fermi problems encourage persistent involvement of students in the solving procedure, useful from the child's perspective and involving basic mathematical skills. Haberzettl et al. (2018) worked on the extent to which Fermi problems can help to expand children's modelling skills. They documented a sequence of lessons that provided an insight into the implementation of modelling tasks in the classroom. All these studies have a similar perspective: they study the work of each group but do not consider the individual contributions of each student.

Group work and interaction in the mathematics classroom

In his analysis of the modelling processes among high school students when solving a Fermi problem, Ärlebäck (2009) highlighted the importance of social relationships in group work. According to Blatchford et al. (2003), group work is 'pupils working together as a team' (p. 155). This should involve children as co-learners (Zajac & Hartup, 1997). Blatchford et al. (2003) explain that in group work 'the teacher might be involved on various occasions, but the key aspect is that the balance of ownership and control of the work shifts towards the students themselves' (p.155). However, the exploration of teachers' experiences has revealed that they feel they lose control, with greater disruption in the class and an increase in off-task behaviours, and with these factors being the main reasons for avoiding work in groups in the classroom (Cohen, 1994). This attitude to group work on the part of some teachers, as described by Cohen (1994), is at odds with proposals for mathematical modelling work. Zawojewski et al. (2003) suggest that students—working in small groups and tackling a problem situation that is meaningful and relevant to them—will invent, extend and refine their own mathematical constructions to meet the demands of the modelling problem. So, there is a need for structuring modelling activities and proper preparation of the students in order to carry them out.

Many aspects demand consideration when preparing a classroom group activity. Not only is what happens during the group work important but also the preparation prior to this activity. According to Blatchford et al. (2003), there is considerable disparity between the potential of group work to influence learning, motivation and attitudes towards learning and relations and its actual—limited—use in schools. During group work, there are many additional aspects that can influence student interaction, given that one of the key aspects is working with heterogeneous groups. A great deal of research has been carried out to identify and achieve the *perfect* composition of a group in order to accomplish the goals set by teachers. The following relevant aspects have been identified by research: group size (Webb et al., 1997), students' individual abilities (Armstrong, 2008), gender (Jiang et al., 2017; Perrenet & Terwel, 1997) and student's personality and identity (Bishop, 2012). The relationship between students (Newcomb & Bagwell, 1995; Strough et al., 2001) has proven to be a key factor since those students who are better socially accepted and have more friends among their peers are those who benefit most from group work and made greater gains in mathematical problem-solving (Klang et al., 2021).

Goos et al. (2002) state that collaboration among peers provides a way of learning about other students' reasoning, their points of view, their ways of solving a problem and the different interpretations given to a particular task. In these situations, students propose, defend, clarify and justify their ideas in front of their peers and conflicts arise as a consequence, which have to be negotiated to reach agreements. During group work, students interact to achieve a certain goal established by the task they have been set. There are many areas of group behaviour open to study and analysis. These include students' engagement and participation, their socio-emotional attitudes, student–student dialogue and sustained discussion on the topic (Baines et al., 2009). In this study, we focused on students' engagement and participation during social interaction when working in groups.

To do so, we took Bishop's work (2012) as a basis for our study. She analysed the interaction between two students who worked together during a mathematics unit. She focused on the girls' identities and how they developed during the sessions in which they worked together. She drew up a series of categories to characterise each student's interventions in order to describe what was being accomplished in each case and identify possible patterns. In our research, this framework helped to characterise students' interventions during group work.

Methodology

The methodological framework of this study was based on the qualitative-interpretative research paradigm. According to Cohen et al. (2000), the characteristics that justify this choice of methodology are that the goal of the research is to interpret a specific phenomenon and understand the students' actions and their meaning. Furthermore, the instruments and strategies it uses for data collection are flexible and can be adapted to the evolution of the research, being open to contingencies.

Participants

The data collected in this study were produced by sixth-grade students completing Primary Education at a state school in Santa Coloma de Gramenet (Barcelona, Spain). The class group consisted of 21 students (12 girls and 9 boys) aged 11 to 12. This was a convenience sample as the teacher of this class group had previously carried out research linked to Fermi problems. Therefore, he already knew how to work with them and was familiar with the data collection process.

Fermi problem used in the activity

The question we asked students was the following Fermi problem: *How many people can fit in the schoolyard?* In this case, students had to carry out an estimation of the number of people who could be organised to fit in a given surface area. We decided on this problem because it has been used in several previous studies (Ferrando & Albarracín, 2021; Ferrando & Segura, 2020). Therefore, we knew what solving strategies the students might use and the mathematical models they might create. Moreover, another aspect we took into account was the students' familiarity with the area. Precisely for this reason the problem was set in the students' school. It is of paramount importance for students in early educational stages tackling this type of problem to be familiar with the area under study so that they can experiment, make measurements, imagine and understand the situation, etc.

To contextualise the problem, we told students we wanted to organise a party for the end of the school year. Therefore, it was essential for us to know how many people could fit in the playground before sending out the invitations. The school had already organised this type of event and so the situation presented was familiar to the students and easy for them to understand.

Data collection

The data collection activity took place during class time, using the natural groups the students already belonged to. These were heterogeneous groups, with students possessing different levels of mathematics. The children in this class were used to working in groups and they all knew each other. In fact, the class was already divided into five different groups.

The first step was to present the problem they would have to solve. For this purpose, we gave each student a printed copy of the worksheet with the problem statement, and we read it out loud several times. We highlighted the fact that we were asking them to explain the strategy they would use to solve the problem (Albarracín & Gorgorió, 2014). The teacher gave no further information to students to help them solve the problem. From here on, he only took part in the activity when it was time to move on to a different stage—for example, when going out to the playground.

The activity proposed in the classroom consisted of five stages as shown in Table 1. The first step was to present the Fermi problem students would work on. We gave time to the children to work individually on it and propose a possible strategy to solve it. In the following stage, students were divided, when possible, into groups of four (Webb et al., 1997) and worked together to develop a mathematical model to solve the Fermi problem. This approach is consistent with the findings of Li and Goos (2021) who observed that groups that have discussions prior to solving the problem obtain better results. This is the stage that we focused on in this study (marked with an asterisk in Table 1). These discussions were the centre of attention of the study and were recorded for later analysis. After this first discussion, students filled in a group worksheet. Once all the groups had decided what concepts and

Table 1 Stages of the activity and type of data collected

STAGE	Presenting the problem and individual work	**Working in groups to develop a model***	Working in groups in the playground**	Sharing the results with all the class	Final individual reflection
DATA COLLECTED	Individual sheet 1	Group sheet *(first time)* Groups' discussions recorded	Some group discussions recorded	Group sheet *(second time)* Whole class discussion recorded	Individual sheet 2

procedures they needed to solve the problem, they carried out the task in the school playground. The discussions in the field were also recorded and used in the analysis, but only in the cases where students re-worked the models they had created (marked with a double asterisk in Table 1). To finish the activity, we asked each group to share their strategies, procedures and results with the rest of the class.

When students made a decision on a procedure that enabled them to come closer to the solution to the problem, this decision was expressed in writing, either by way of a summary, a sentence or a specific calculation. In this way, the mathematical models generated by solving the problem could be characterised (Lesh & Harel, 2003). However, the students did not take notes on the different ideas they considered during their discussions and so we recorded their conversations on audio in order to observe their negotiation processes. For this reason, both oral and written data were taken into account when answering the questions and accomplishing the goals set previously. As regards the oral data, this included the recordings of the students' interactions during the model development stage and while working on the playground stages. On the other hand, the written data produced throughout the activity had a secondary function. It consisted of students' reports before, during and after the problem-solving process. Thanks to this, data triangulation was possible and provided us with a complete view of the phenomenon.

Data analysis

After data collection, the recordings from the model development stage were transcribed and structured in order to simplify the categorisation of the students' interventions during the group work. To study the strategy negotiation process and understand what factors influenced the group work, we decided to analyse the data from three different perspectives. We first looked at the progress of students' ideas during group work, identifying who introduced each of these ideas. Next, we categorised the students according to their interventions. And finally, we listed the factors that might influence group work but without going into any detail. In order to understand some of the results, we also interviewed the class teacher, who described each student's social and academic skills.

First of all, to explore the progression of mathematical ideas, we focused on the models and strategies proposed by students and discussed during the group work. We identified the models proposed individually by each student. To do so, we drew on the mathematical model characterisation established by Gallart et al. (2017) for the same problem. In this study, the authors proposed the following categories to identify students' models:

- *Reduction and use of proportion:* solving an equivalent problem with smaller values and using a proportion factor to contrast the two situations.
- *Concentration measures:* ascertaining the number of people or objects in a portion of a given surface area determined by the students.
- *Reference point:* determining the total surface area where people could be placed and dividing it by the surface area that one single object occupies. This acts as a

unit that can be used by students, which is called a reference point (Joram et al., 2005).

- *Grid distribution:* distributing people or objects on a grid and estimating their number for each dimension (for example: height and width on surfaces) and using the product rule to obtain a final answer.

In this research, we also included the possibility of students presenting strategies that were not suitable for a modelling process that solved the problem presented. These were categorised as 'no strategy identified'.

Once we had identified each student's initial model, we followed a timeline and observed all the ideas introduced in relation to mathematics and the problem to be solved. Second, the students' interactions were analysed according to their content. We coded students by group and gender and gave them a number. So, for example, Student AF1 is from group A, and she was the first female analysed. Categorisation of students' interventions was based on Bishop's study (2012) and it analysed student interactions from three perspectives: structure, function of each intervention and content. The aspects of this categorisation that were most relevant to this research were the content and function of students' interventions and the identity of the speaker.

The categories presented by Bishop (2012) to classify students' interactions are as follows: *low level give*, *high-level give*, *low-level request* and *high-level request*. These four categories implicitly include the function of each intervention because they are based on whether the student is giving or requesting information and the mental work they require (Fig. 1). We decided to add another category: not relevant. This served for comments that did refer to the problem being solved or the group work. Each intervention was coded according to its function, content and the speaker (Bishop, 2012).

The following extracts in Table 2 are examples of student interventions and how they were codified using the categories described above:

Following data collection, only four groups were analysed using this methodology. The reason for this was the lack of interaction among the members of the fifth group. As the main focus of this research was to analyse interaction among peers and how they work together to develop a model to solve a Fermi problem, the absence of interaction made their work irrelevant to the research. When the teacher was asked about the group, he explained that it consisted of very shy, introvert students and this was the main reason for their lack of participation.

Results

As explained above, the analysis of the group work focused on two main aspects: student interaction and the mathematical work done in each group. Table 3 shows the results of the four groups analysed. It considers the models presented, who described them and student interaction.

An important aspect of group work is the progression of the ideas that have been introduced and how these are developed by students. The following figures show

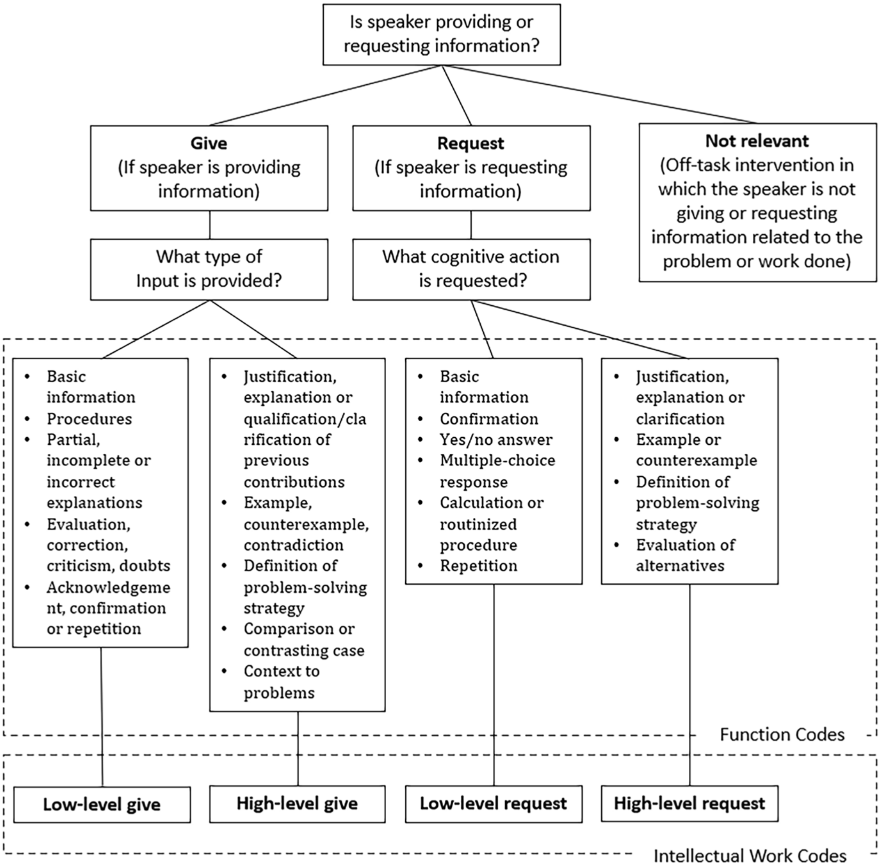

Fig. 1 Flowchart of function codes and mental work codes adapted from Bishop (2012)

Table 2 Student intervention and categories applied in analysis

Student intervention	Category
M6T: què?… mmm… què estratègia és la teva? (what?… hmm… Which strategy is yours?)	High-level request
F9V: no a ver espera, si en les dues àrees que hi ha a cada punta del camp ja sumen 54, entre 27 i 27 ens hem d'imaginar més àrees i les hem d'omplir amb 27 més. Que més àrees seria… 27 més 27 més 27 més 27 (No, wait! If the two areas at each end of the playground total 54, which is 27 plus 27, then we have to imagine more areas and fill them with more 27 s. With more areas, it would be… 27 plus 27 plus 27 plus 27)	High-level give
F3B: creieu que està bé? (Do you think it's right?)	Low-level request
M4B: si, jo crec que si (Yes, I think so)	Low-level give

Table 3 Characterisation of group work

	Group A	Group B	Group C	Group D
Group composition	4 students (three girls and one boy)	4 students (one girl and three boys)	4 students (three girls and one boy)	5 students (three girls and two boys)
Models used (students who introduced it)	Reduction and use of proportion (AF2) Grid distribution (AM4)	Reduction and use of proportion (BM1) Grid distribution (BM1)	Reduction and use of proportion (CF2)	Reference unit (DM5) Reduction and use of proportion (DM4) and grid distribution (DM4)
Group interaction	293 total interactions: HG 60, HR 16, LG 159, LR 46	125 total interactions: HG 28, HR 7, LG 53, LR 22	229 total interventions: HG: 24, HR: 2, LG: 177, LR: 13	206 total interactions: HG 30, HR 8, LG 143, LR 24
Student participation	AM4 (100), AF2 (99), AF1 (85), AF3 (9)	BM3 (44), BM1 (35), BM4 (31), BM2 (15)	CF2 (72), CM4 (66), CF3 (62), CF1 (29)	DM4 (71), DM5 (62), DF3 (51), DF2 (21), DF1 (1)
High-level give interventions	Mainly student AF2 (30), followed by AM4 (16) and AF1 (12)	Students BM1 (9) and BF4 (9)	Student CF2 (12) followed by CF3 (7)	Students DM4 (12) and DM5 (9)
Other relevant aspects	AF1: 22 interventions requesting information. AF3: She only participated when asked by her peers	BM1: During group work he imposed his ideas, not listening to his peers. The other members of the group collaborated, while he refused to participate	CM4: He participated a lot but mostly with low level (59) or off-task (4) interventions	DF1: She did not participate. DF3, DM4 and DM5: They did most of the work while ignoring their peers. DF2: She mostly made low-level interventions (17)

each group's evolution, taking into account the arguments and aspects debated during group work and which student introduced each of them. Also, there is a brief description of the progression of mathematical ideas in each group.

As regards the coding of students, the first letter represents the group they were in, the second letter refers to gender and finally, the numbers, assigned randomly within the group, refer to specific students. The coding of the interventions was as follows: HG (high level give), LG (low level give), HR (high level request) and LR (low level request).

Progression of mathematical ideas: group A

As shown in Fig. 2, the first step taken by group A was to share the individual strategies, even in those cases where the strategy presented was not suited to solving the problem. Then they tried to estimate how many people would fit in the playground. However, they ended up solving a different problem to the one proposed initially: instead, they proposed a strategy to find out how many people could come to the party. When comparing strategies with their classmates and the teacher, they realized they had misunderstood the problem. In the playground, they started all over again, introducing several ideas about two models to solve the Fermi problem:

- *Reduction and use of proportion:* Student AF2 suggested using a picture from la Marató as a reference (a charity act held at the school, with many people). Her idea was that if they knew how many people there were in the photograph, they could multiply this number to fill the whole playground.
- *Grid distribution:* Student AM4 introduced the idea of counting how many people could be placed in a line on each side of the playground and multiplying the totals. So, their next step was to count by positioning themselves one next to the

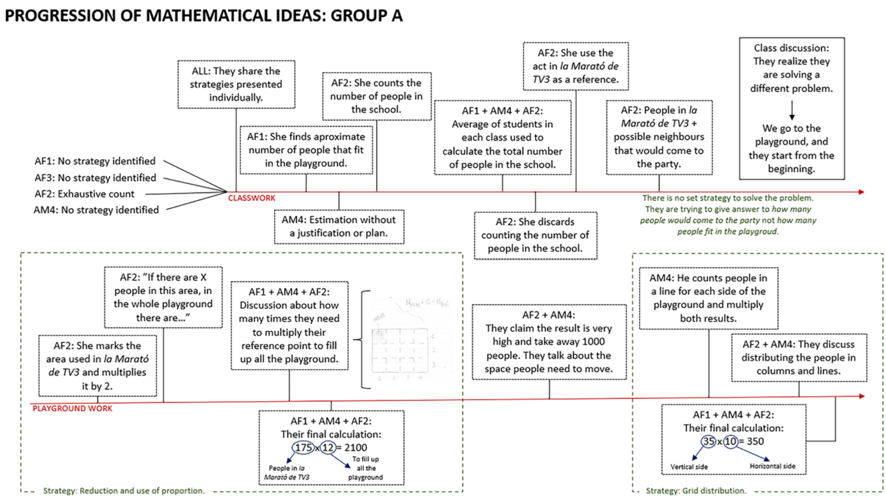

Fig. 2 Progression of mathematical ideas. Group A

other until they completed the long side of the playground and then to do the same for the short side.

Progression of mathematical ideas: group B

First, all the members of group B (Fig. 3) shared their individual proposals, including the student who proposed a strategy that was unsuited to solving the problem. In this case, Student BM3 did not provide a clear strategy and directly affirmed that his answer was wrong after listening to the other group members. Furthermore, Students BM2 and BF4 explained procedures that solved a different problem to the one proposed initially, showing that they had misunderstood the problem. During the first part of the group discussion, they focused their attention on whether they should calculate the length of the playground or count the number of people who could come to the party. As a result, the two main models the group worked on were as follows:

- *Grid distribution:* This was the first strategy proposed by BM1, and the one he presented individually. He explained it clearly to his peers, even with a drawing. He insisted on the need to calculate the length of the playground.
- *Reduction and use of proportion:* This model was introduced but not developed during the recorded group work because BM1 suggested it at the end. However, they described the procedure in the reports they had to hand in at the end of the session. They proposed using themselves as a reference. They explained that in one of their physical education activities, they all had to gather in the small area of the football field. And so, they multiplied this area by three to fill in half of the playground. And they multiplied it again by two to fill in all the rest.

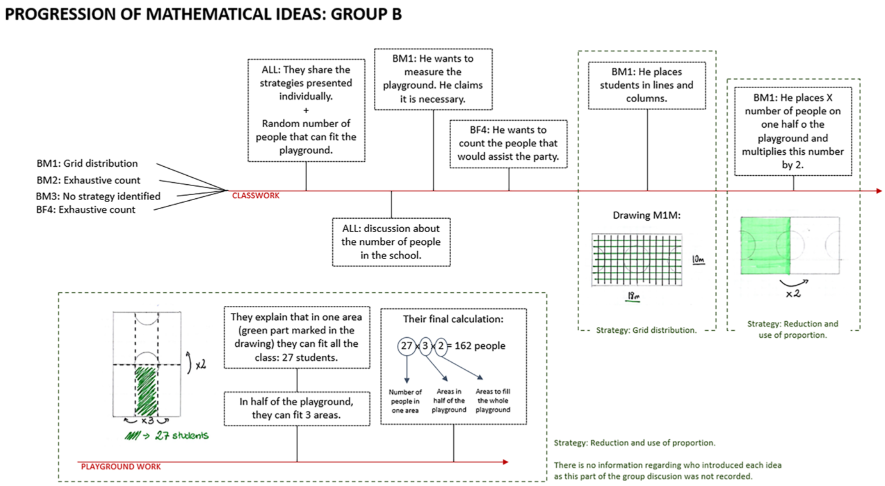

Fig. 3 Progression of mathematical ideas. Group B

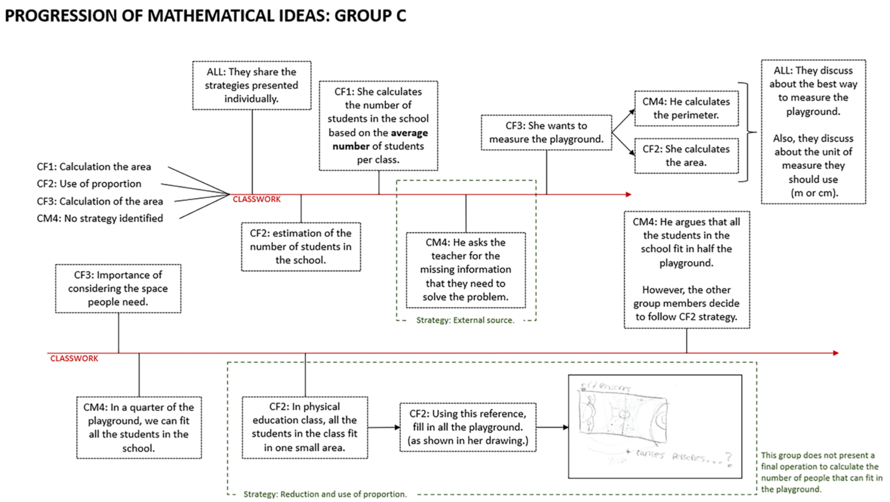

Fig. 4 Progression of mathematical ideas. Group C

Progression of mathematical ideas: group C

The first step taken by group C (Fig. 4) was to share the strategies proposed individually, even in those cases where the students suggested an unsuitable strategy to solve the problem. They listened to each other and decided to combine all the strategies to find one final model that would serve to solve the problem. All the members of the group identified the dimensions of the playground as essential information. This resulted in a discussion of how to measure the playground and which variables should be considered. Their first instinct was to ask the teacher if he could help them. After this they started working on the main model:

- *Reduction and use of proportion:* CM4 introduced the idea that they could fit all the students in the school in a quarter of the playground. Then, CF2 explained that in physical education class they used a smaller area. She continued with this idea and made a drawing to explain how they could fill in the whole playground using this reference. The other group members understood her idea and they examined it for possible mistakes. They pointed out that the area she wanted to use as a reference was a semi-circle and that when filling in the whole playground, some of the space would not be included. However, they concluded that they were only looking for an approximate result.

Progression of mathematical ideas: group D

The students in group D (Fig. 5) started out by sharing all their individual strategies, and even those students who had a strategy that was unsuited to solving the problem participated. However, they did not discuss these proposals. Instead, they started to calculate directly until the teacher reminded them that they only had to look for the steps needed to solve the problem. Then they decided to

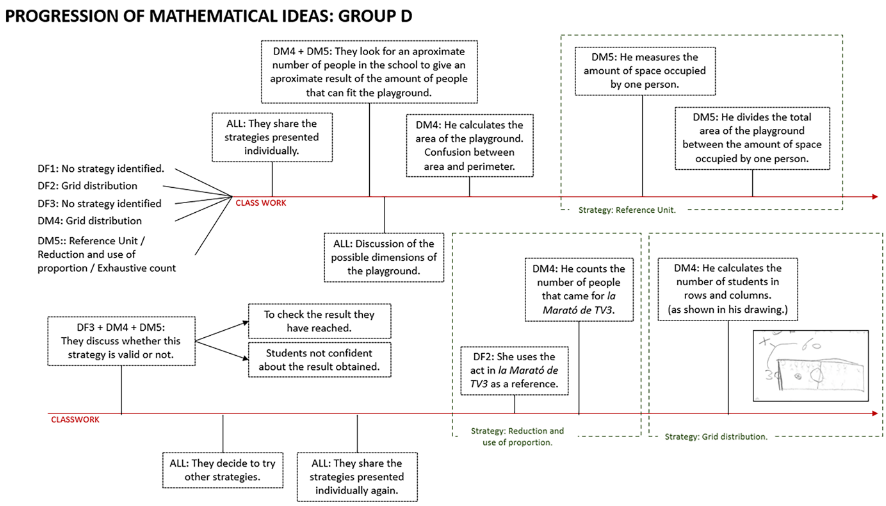

Fig. 5 Progression of mathematical ideas. Group D

start looking for a suitable strategy to solve the problem. At the beginning of the group work in particular, only Students DM4 and DM5 participated actively. The three models this group proposed to solve the problem were as follows:

- *Reference unit:* Student DM5 suggested measuring the amount of space occupied by one person and then dividing the total playground area by the result to find out approximately how many people in total could fit in it. They carried out this strategy using measurements they were not able to calculate at the time. Once they found the result, they noticed the number was very high and they did not trust it. Therefore, they decided to look for another strategy, to check the result and find out whether there was an easier way of solving the problem.
- *Reduction and use of proportion:* After going back to the beginning and sharing their strategies again, they started estimating the number of people who could fit in the playground based on the number of students in the school. Student DM4 then realised they could use an event such as 'la Marató' to calculate the number of people that would fit in the playground. They did not explain why they later discarded this strategy.
- *Grid distribution:* This was introduced by student DM4. He proposed distributing the students in rows and columns, as shown in his drawing (see progression of mathematical ideas above). The whole group decided this was easy to put in practice because when they went down to the playground, they would be able to calculate the number of students that fit in each row and each column. Therefore, this was the final model they used to obtain an answer to the problem.

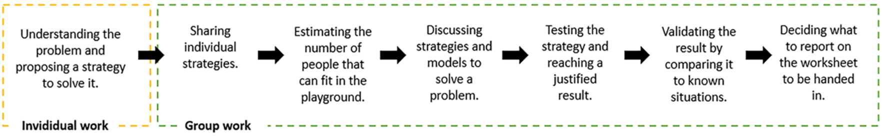

Fig. 6 Steps taken to solve a Fermi problem in groups

Discussion and conclusions

Many aspects of the modelling process emerged during the group work. As the problem was set in a known space, students easily related the problem-solving process to the real world. As a result, some aspects such as people's needs—the need for space to move around in, for example—were considered as well as the viability of the strategies (Albarracín & Gorgorió, 2014). Another point is that there was a constant push to calculate and provide numerical information. All the groups wanted to test out the strategies straight away, even though the statement of the problem specifically asked them to only find the steps needed to solve it. This may be related to the fact that in most mathematics activities, students are used to solving operations and only give any importance to the results obtained.

Like in previous studies, the working groups generated mathematical models to solve the Fermi problem posed (Albarracín & Gorgorió, 2018; Peter-Koop, 2009). The same as in a previous study by Ferrando and Albarracín (2021), the reduction and use of proportion model was introduced by all the groups but only two of them used it as a final model. The grid distribution model was also taken into consideration by three groups and used as the definitive one on two occasions. Finally, one group also discussed using a reference unit to solve the problem. However, this only received a passing mention and was not put into practice. The changes that occurred during the group work show that students linked their own concepts and procedures to those proposed by the other group members in order to shape their mathematical model (Sevinç, 2021). In these circumstances, the pupils have to cope with a conflict, which is that everyone is trying to solve the same problem but not everyone has thought of the same way of solving it. This aspect was reinforced by the inclusion in the activity of a time slot during which students were asked to develop an action plan individually. When students generate a mathematical model collaboratively, they connect and coordinate different elements, whether conceptual or procedural (Lesh & Harel, 2003), and it is possible that even when proposing different models, some of these elements are shared or can be adapted to the model being constructed.

The goal of this study was to characterise the exchange of leading ideas during the negotiation processes that lead to the construction of a mathematical model through student interaction while solving a Fermi problem in groups. Figure 6 shows the path taken by all groups when developing a model to solve a Fermi problem. Not all the groups spent the same amount of time discussing the strategies or the results, but they all followed the same path in the end.

All the groups started out by sharing their individual strategies, even the students with a lower level of mathematics and those who did not participate in the rest of the group work. Afterwards, they tried to estimate the approximate number of people that could fit in the playground. While doing so, they engaged in a discussion where many aspects of mathematics were discussed (Zawojewski et al., 2003), for instance, concepts of perimeter and area and the unit of measure. At this point, at least one student stopped participating actively. Then, they tested out the results and compared the numbers to those of previous events held in the same space. Finally, they summarized what they had done and decided what to report on the group worksheet.

When starting the group discussion and sharing the strategies proposed individually with their peers, in most of the cases these were discussed and provided the basis of the group work (Goos et al., 2002; Li & Goos, 2021). However, in two groups they ignored some of the explanations presented by other students and immediately decided on a certain strategy. This left the impression that they had only shared all the strategies because 'it was the right thing to do' when working with their peers. The analysis carried out does not provide any way of knowing why this happened. On the other hand, at the end of the activity, all the groups tried to reach a full consensus when deciding what to report on the final worksheet.

As far as student participation is concerned, the students who involved themselves the most in groups A, C and D were the ones who introduced the ideas that evolved into the final models. This was also observed in the high-level give interventions, where the same students had the highest number of interventions in this category. This behaviour is coherent with the findings of Ng (2008), which showed that leaders of successful groups who were socially non-dominant but mathematically active were more likely to apply a higher frequency of basic thinking skills than group members in other roles. On the other hand, group B did not follow this pattern completely, as the student introducing the main ideas was not the one who participated the most. This can be explained by the fact that some students were more willing than others to collaborate, to provide input for discussion and to adapt their ideas to the model being developed by the group. In this study, we cannot confirm whether this was specifically due to individual personality traits and the pupils' identity (Bishop, 2012) or aspects related to their relationships (Klang et al., 2021; Newcomb & Bagwell, 1995; Strough et al., 2001). What we can conclude is that group behaviour was highly dependent on the willingness of its members to cooperate.

As regards the type of interventions made by students, the vast majority were low-level give in all groups, which more or less concurs with the results of Bishop's work (2012). Furthermore, the high-level give interventions were explanations or procedures, with students rarely, for instance, giving examples. Regarding low-level and high-level request interventions, these were rather infrequent compared to low-level and high-level give. This result also tallies with Bishop's study (2012) since most of the student interventions required a low level of mental work.

In all the groups, the members who participated actively worked together to improve their models and search for the missing information. In other words, there were some students who stood out in each group, but other members of the same group also participated and made suggestions to develop the proposed model. It was through this peer interaction that the students identified the possible weaknesses of their strategies. This was a process done collaboratively rather than by questioning each other's ideas. The number of student interventions where information was requested of other group members was considerably lower than the number of interventions giving information.

Thus, this exploratory study gave us the opportunity to observe a phenomenon and identify several aspects that may influence the outcomes of group work during a modelling task. Based on these findings, we consider that it may be necessary to introduce mechanisms into the design of the activity that ensure both the exchange of ideas and the participation of those students who are less involved. This could serve as a pedagogical strategy giving greater control to teachers in open activities (Cohen, 1994). These mechanisms would be based on the process description presented in Fig. 6. A modelling activity allows for a variety of approaches from the standpoint of teacher intervention. Students can be allowed to act freely with the sole objective of solving the problem, or the task can be structured to facilitate the students' work in a scaffolding mode. Moreover, understanding the progression of group work during the solving process of a Fermi problem has various implications when designing and implementing this collaborative learning environment. For example, knowing each student's role and attitude during group work is essential because these are core factors that influence group work. Also, it is important to be aware of the difficulties students will encounter so that teachers can provide the tools needed to interact equitably and work productively in a positive environment.

All groups followed the same path when developing a model to solve a Fermi problem. They shared their individual strategies, estimated the possible result, discussed the strategies used, validated the results, and decided what to write down to present and share with the rest of the class. As a general rule, one student introduced an idea and through social interaction with the other group members, this was developed and improved to build a solid strategy that could serve to solve the problem presented. The specific mechanism by which mathematical models are developed in groups of primary school students seems to consist of considering as a basis the concepts and procedures provided by a member of the group who presents an initial feasible model. On this basis, the work with the rest of the students leads to the addition of new elements that are compatible with the initial model, the intention being to enrich it so that it better describes the situation under study. However, other models can also be introduced and developed by other students or by the same student who proposed the first idea. One student might introduce the initial model, but it is through group work and student interaction that the definitive mathematical models are developed. One key finding of this research is that we cannot be sure that the model constructed by the group is understood in the same way by all students. It is reasonable to assume that those students who participate more intensively and make high-level contributions learn to generate or use that mathematical model, but we have no evidence that students who participate to a lesser degree do so. Therefore, we consider it necessary to explore the way each student is encouraged to make his/her contributions to the mathematical model constructed during the activity and whether he/she has the possibility of understanding each procedure proposed by his/her group partners afterwards.

In view of the fact that we only analysed the group work of students in one class group, the main limitation of this research is the amount of data collected. Also, some aspects of the context may have influenced the results obtained, for instance the students' background, the class teacher and the school environment in general.

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Declarations

Ethics statement The authors confirmed that all original research procedures were consistent with the principles of the research ethics published by Universitat Autònoma de Barcelona in its code of good practices in research (https://www.uab.cat/doc/DOC_CBP_EN). Universitat Autònoma de Barcelona does not require ethical approval to conduct educational research.

Conflict of interest The authors declare no competing interests.

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