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ORIGINAL PAPER

Mathematical naming and explaining in teaching talk: Noticing work with two groups of mathematics teachers

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Abstract
Research shows the salient place of mathematical teaching talk, including the mathematical-linguistic practices of naming and explaining, in the enactment of students' mathematical talk and learning with understanding in the classroom. Our study was developed to examine the noticing of two groups of secondary-school mathematics teachers in one-day workshops with tasks about these practices. The two workshops were mathematically content-specific, with teaching and learning accounts and prompts aimed at guiding focused attention to naming and explaining in the teaching of linear equations and probability. Thematic text analyses led to identify three foci of the two groups' noticing: (i) missing practices of mathematical naming in own teaching talk; (ii) relative impact of mathematical explaining in teaching talk; and (iii) tensions around mathematical naming and explaining in teaching talk. Our results show that the social construction of teacher noticing is a feature of noticing development that can be documented in the context of one-day workshops. Whereas time for individual thinking and responses to the tasks created a context of support for noticing development, participation in the group discussions allowed the teachers to notice nuances of mathematical naming and explaining in teaching talk unaddressed in the task prompts. The group discussions thus amplified and opened up the opportunities to develop some focused noticing on the content of the workshops, specifically in connection with the teachers' own teaching practice.

1 Introduction

Talk is a crucial communicative activity in classrooms, including the talk of the teacher in the interaction with the students, with their talk and ways of reasoning. Hence, understanding mathematics teaching and learning entails understanding the talk that is developed in teaching. In the study of mathematics teaching, nonetheless, teacher talk is often subsumed within the study of other important aspects of teaching such as the mathematical tasks chosen (Lampert, 2001). Moreover, when the study of teacher talk in teaching is in focus, the mathematical-linguistic practices at the smaller levels of words, phrases and sentences tend to be subsumed within or subordinated to the study of mathematical-linguistic practices at the level of larger utterances (Longwe et al., 2022). Some mathematical practices at the smaller levels of language are for naming other words, symbols, images, concepts, procedures or relationships in mathematics (Adler, 2021), or for explaining, which implies a focus on meaning and on giving reasons (Ingram et al., 2019). Despite all this, developmental initiatives on the linguistic practices of mathematics teaching remain scarce (Planas et al., 2023).

A shared interest in mathematics teaching talk led us to collaborate in a project started in 2020 with an agenda of one-day workshops for secondary-school mathematics teachers. All the workshops are based on tasks to support noticing work around mathematical naming and explaining in teaching talk, in ways that respect and challenge the teachers' pedagogical knowledge. Given the paucity of attention to the mathematical-linguistic practices of naming and explaining in our educational contexts, we had anticipated that the teachers would struggle with the workshop tasks. However, we came across vivid and focused discussions in a group of seven teachers (Group 1) in three workshops on naming and explaining in the teaching of linear equations, fractions and plane isometries to enact students' mathematical talk and reasoning (Planas & Alfonso, 2023). The idea of the current study emerged from a seminar on those discussions.

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At that time, data from a second group (Group 2) had been collected, in a workshop on naming and explaining in the teaching of probability (WS2). We decided to explore further nuances in the Group 1 noticing during the workshop on linear equations (WS1), and to examine whether these could be also representative of the Group 2 noticing. In this paper, we discuss the data from Group 1 in WS1 and Group 2 in WS2.

2 Literature review on the sociocultural trend in mathematics teacher noticing

In their survey paper for the 2019–2022 research literature on teacher noticing in mathematics education, Weyers et al. (2023) concluded about the dominance of psychological-cognitive orientations and the pervasiveness of the focus on students' mathematical thinking. These authors also concluded about the increasing diversification of foci and theoretical orientations in the literature surveyed, including social and sociocultural orientations. The survey by König et al. (2022) already documented the relative increase of sociocultural perspectives of mathematics teacher noticing with studies that situate the teacher within group activity and noticing as social and interactional. This growing attention to the social practice of noticing aligns with what Dindyal et al. (2021) and Amador and Weston (2024) also commented in their surveys. As it happens with other trends in mathematics education research on teacher noticing, the trend that builds on sociocultural perspectives is highly diverse. The connection with the notion of professional vision in Goodwin (1994) is, however, common; noticing is not psychological, it consists of "socially organized ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group" (p. 606).

Within the sociocultural trend of research in mathematics teacher noticing, it is assumed that there is value in teachers working together, and that this working serves as a mediator for noticing development. Some of the studies, nonetheless, keep their primary analyses and results at the individual level. With a focus of noticing on equity aspects of mathematics teaching, Crespo et al. (2021) analysed the individual noticing statements of student teachers when interacting with learners from underrepresented groups in simulated scenarios of mathematics lessons. Other studies integrate analyses and results at the individual and group levels to examine the social construction of noticing. Amador et al. (2023) analysed individual noticing in video club participation and moves towards collective noticing in joint discussion of selected videos. These authors studied the role of individual teacher noticing and of group activity in the social construction of noticing, and considered the discussions in the video club as evidence of collective noticing development. Research into the opportunities of noticing development that group discussions may enact as teachers interact with one another is still scarce in mathematics education. Even when the educational setting involves group discussions, it is common to select and examine data from written accounts of individual teachers.

Within the sociocultural trend of research that primarily attends to the social construction of mathematics teacher noticing, some of the studies build on dialectical stances. Close to the ideas of the "third space" in Williams and Ryan (2013, p. 210), or to the "surplus of seeing" in Bakhtin (1990, p. 134), these studies see the potential of two or more people engaged in discussion to bring new seeing directions regarding what to attend to and how to signify it. Barnes and Solomon (2013) discussed mathematics teacher noticing as dialectically happening in-between the experiences of the researchers, the teacher educator, and the interactional settings of participation. In the collaboration with four mathematics teachers, van Es et al. (2017) illustrated the "third space" generated at the intersection of the teachers' experiences of equitable teaching practice and the researchers' seeing of this practice and of the developmental work for reflection on equitable teaching. These two studies emphasised the dialectical and co-produced nature of mathematics teachers' noticing in professional development. The discussions of the teachers were key to the generation of a third space in which different teaching experiences, assumptions and knowledge interacted.

3 Conceptual framework for teacher noticing with a focus on aspects of mathematics teaching talk

In this section, we introduce the main notions of our framework. Elsewhere (Planas & Alfonso, 2023), we presented the notion of students' content-specific learning challenges that guides the task design in the workshops, and which is particularly important in the attention to results of the teachers' noticing that are specific to the mathematical content in focus.

3.1 Mathematics teaching talk, mathematical naming and mathematical explaining

Pimm and Sinclair (2009) claimed that "the shaping of form by content and of content by form" (p. 24) is a fundamental question of mathematics teaching and learning. A particular instance is the spoken form of mathematics teaching talk, that is, the mathematical talk of the teacher with the students in the classroom. Like Pimm (1987/2017), we understand mathematical talk as a subset of the linguistic notion of the mathematics register in Halliday (1975), and hence *mathematics teaching talk* as the oral use of the mathematics register by the teacher in classroom teaching. Regarding this talk and what makes it different from other subject talk, Lampert (2001, p. 152) referred to small utterances consisting of individual words (e.g., “times”), phrases (e.g., “ten times twelve”) and sentences (e.g., “twelve times ten equals ten times twelve”). O’Connor and Michaels (2019) referred to sentences in teaching intended to enact mathematical discussions (e.g., “what happens if we add another negative number here?”, p. 167). Götze and Baiker (2023) and Ingram et al. (2019) also drew attention to small units of talk in studies of mathematics teaching and the building of mathematical discourse. Our focus on small units of word use in mathematics teaching talk is similarly framed within a notion of mathematical discourse (Moschkovich, 2021) that includes the mathematics register and building practices alongside the diverse forms of communication in mathematics.

We see the choices concerning small utterances in mathematics teaching as basic for supporting the modelling and enacting of mathematical discourse. When words are used in the mathematics classroom, Morgan and Alshwaikh (2012) argued, all those in the interaction respond by developing some participation in the processes of meaning making prompted by those words. A feature of language is this power to enact or hinder participation and meaning by virtue of one sentence, one phrase or even one word (Halliday, 1985), some of which can function for mathematical naming and/or explaining. In our study, the *mathematical naming* of other words or objects such as symbols and images (Adler, 2021) is mathematics teaching talk responding to what-questions and functioning to make explicit or clarify what is being discussed so that students can focus on it. The forms involved are nouns or nominalised words and phrases, used in articulation with other forms of communication in the context of mathematical discourse practices, such as explaining or representing mathematical symbols and diagrams with words from the mathematics register. The linguistic practice of *mathematical explaining*, which implies a focus on meaning and on giving reasons (Ingram et al., 2019), is mathematics teaching talk responding, in varying depth and detail (Evans et al., 2022), to why-questions and consisting of sentences that communicate explanations of mathematical meanings and relationships within the mathematics register. These forms are again used in articulation with other forms of communication in the context of building discourse practices, such as naming or connecting concepts mathematically.

Thus, our notions of mathematical naming and explaining account for more than linguistic form. These are practices interdependent with one another, and with other building practices of the mathematical discourse. They are basic practices on which other mathematical practices, such as proving or generalising, are built, but they are also critical. Teaching talk can bring up explanations with ambiguous pronouns, it can confuse naming and explaining by naming concepts and not explaining them, or explaining and representing with symbols. The notation $(x+1)\cdot(x-1)=x^2-1$, for example, requires naming phrases such as ‘an identity’, in response to the question of what this algebraic expression is, and explaining sentences such as ‘This is not an equation to be solved, but an identity because the equality is always true’, in response to the question of why it is an identity. The naming of ‘equation’, ‘identity’ and ‘equality’ together respond to the question of what concepts related to algebra are in focus in the explanation, which can be a basis for enacting mathematical distinctions and explaining the concepts named. Moreover, any linguistic account of mathematical naming and explaining is situated in a communication context. What is said by the teacher is about what is said and done in the classroom –a student may have said that $(x+1)\cdot(x-1)=x^2-1$ is an equation to be solved– and implies some metacommunication (Pimm, 1987/2017).

3.2 Teacher noticing around mathematical naming and explaining in teaching talk

Naming and explaining can be mathematically correct and precise, yet unresponsive to the students’ learning in a classroom. The teacher may not assess the students’ participation in the mathematical discourse and accordingly use these practices to support their understanding. Linguistic responsiveness in mathematics teaching (Neumayer DePiper et al., 2021) requires, at some stage, a focus on the linguistic practices of mathematical naming and explaining, on how these are related to and a component part of other building practices of the mathematical discourse, on how they interact with diverse forms of communication, and on how they amplify or react to the students’ talk. Some studies have examined work on linguistic responsiveness especially attentive to mathematical naming and/or explaining. The groups of teachers in Longwe et al. (2022), Planas (2021) and Planas et al. (2023) discussed the criticality of utterances of mathematics teaching talk in which the reading of mathematical symbols and symbolic notation did not have a focus on meaning. Several other studies have placed pedagogical emphases on explicit and precise word use. Otten et al. (2019) conducted group sessions with mathematics teachers in the US. One of the tasks was to discuss lesson transcripts and to reflect on the function of explicit teaching talk about quantities, calculations, vocabulary and symbol use in fostering mathematical meaning. Explicitness and precision were associated with modelling mathematical explanations and names in interaction with other practices of the mathematical discourse, such as connecting symbolic, visual and verbal representations. The teachers identified utterances in which, by naming and linguistically connecting names of concepts and processes, between them and with symbols, visuals or diagrams, teaching talk had enacted some modelling of the mathematical discourse.

Guided by a frame of explanatory communication, Adler et al. (2023) reported lesson study work with ten secondary-school mathematics teachers in Malawi in which the teacher educator prompted reflective discussions on "how words are used, and justifications made" (p. 42) for geometry teaching. Spoken and written words, and switches between them, were regarded in relation to mathematical practices that connected, for example, linguistic explanations to visual diagrams, and developmental work included examining ways of naming and explaining mathematical diagrams and symbols. Guided by a frame of mathematical dialogue, Sjöblom et al. (2023) considered teaching talk with students in small-group conversations as an object of research and professional learning in Sweden. In cycles of collaborative work with four secondary-school mathematics teachers, the teacher educator emphasised teaching talk as mediational of classroom discussions, and the teachers decided that, in their classrooms, "they would be aware of and attend to their own use of why-questions and try to use them to initiate the discussions" (p. 520). These studies differ in many respects but are both supported by frames of linguistic responsiveness in mathematics teaching.

For us, in contexts of professional development, teacher responsiveness to mathematical naming and explaining involves *teacher noticing around mathematical naming and explaining in teaching talk*. Noticing work can be a strategy in support of developing some teachers' responsiveness that can ultimately bring improvement in classroom teaching and learning. In the building of professional vision (Goodwin, 1994) to become a responsive teacher in classroom teaching, we see "a collection of [noticing] practices both for living in, and hence learning from, experience, and for informing future practice" (Mason, 2002, p. 29), some of which take place in the interaction with others in developmental work. This noticing is dialectical (Engeström, 2015), mediated by experiences of the others and the world, including educational and teaching experiences that shape what the teachers "are sensitized to notice" (Mason, 2016, p. 224) in "ways that are ambitious" (Louie, 2018, p. 55) and that can go beyond what they expect and are trained to see (Sherin & Star, 2011).

Concurrently to the understanding that any noticing is co-produced and develops dialectically in the relation with others, we use a three-layered continuum model of practices of identifying, interpreting and deciding (van Es & Sherin, 2002) to support the design and enactment of focused noticing. In our study, teachers' noticing develops in task-based workshops, whose design is supported by a version of the model in van Es and Sherin (2002). The modified version is a tool in the design of task prompts with potential to enact the teachers' noticing around mathematical-linguistic practices of naming and explaining in teaching talk. As shown in the next section, we specify task prompts aimed at identifying mathematical naming and explaining in utterances of content teaching talk; justifying relationships between students' content learning challenges and utterances of mathematical naming and explaining; and improving or creating utterances of mathematical naming and explaining with potential to support students' content learning. By using aspects of the model of identifying, interpreting and deciding, we do not mean that the teachers' noticing can be reduced into some computation of processes. The adoption of prompting strategies can enact focused attention to aspects of mathematical naming and explaining and, at the same time, can allow the teachers' noticing to develop fruitfully in many other diverse directions.

The diversity of directions in which the teachers' noticing can take place in group discussions, even when prompting strategies are provided in a given direction, leads to our research question: *What do the groups of mathematics teachers notice during their discussion of tasks on mathematical naming and explaining in content-specific teaching talk?* The attention to the group level is guided by our drawing on the sociocultural theory of human activity in Engeström (2015). We approach the discussions in Groups 1 and 2 as expressions of collective noticing in which the teachers, due to their interaction with others, have the opportunity to consider and delve into a wider range of possibilities for what is important.

4 Methods

Empirically, the study consists of two task-based workshops. The application of some aspects of design-based research (Cobb et al., 2003) to earlier pilots with different groups of mathematics teachers (see Planas, 2021, for WS1; Alfonso, 2022, for WS2) helped a great deal with the preparation of the current workshops and the validation of the tasks. In this section, we illustrate the kind of tasks that the teachers of Groups 1 and 2 were asked to discuss, and the role of the teacher educators in bringing up instances of mathematical naming and explaining specific for linear equations (Group 1, WS1) and probability (Group 2, WS2). We then explain the deductive-inductive hybrid methods applied to see directions of the teachers' noticing in the group elaboration of responses to the task prompts.

4.1 Design and implementation of the workshops

We worked with two groups of seven secondary-school mathematics teachers in Barcelona, Catalonia, Spain, with no common teachers in both groups. For the development of the 2021 workshop on linear equations (WS1, Group 1) and the 2023 workshop on probability (WS2, Group 2), approval was granted by the research ethics committee of the first two authors' university. All the teachers held mathematics or science university degrees, had some years of mathematics teaching experience and an interest in improving their classroom teaching. During the recruitment of volunteers, we presented the workshops as professional development sites for discussion of content-specific mathematics teaching talk with a focus on practices of mathematical naming and explaining. The teachers who volunteered claimed not to have been involved before in work on mathematics teaching talk. Two of them stated that they had talked about linguistic practices in a specialised course on CLIL –Content and Language Integrated Learning– pedagogies for teaching mathematics in English, but with focused attention to general talk moves rather than mathematically content-specific practices.

Many dimensions are not comparable between the two workshops in this paper. The teacher educator was not the same –the first author for WS1 and the second author for WS2– and each workshop had a number of tasks and time length. The dynamics and the task-based structure were, however, similar. The teacher educator took 30 min to present i) a selection of secondary-school students' learning challenges from the research literature (e.g., algebra structure sense in the learning of linear equations, and representativeness in the learning of probability), and ii) practices of mathematical naming and explaining by posing and answering what- and why-questions with the potential to address some of the challenges presented (for the representativeness heuristic, Batanero et al., 2016, e.g., "The throws are independent, so all sequences have an equally likely chance of occurring" in response to "Why is any sequence of heads and tails representative in the coin-throwing experiment?"). After the introduction by the teacher educator, the afternoon was for individual thinking and writing on tasks followed by group discussion. Each task sheet included classroom teaching and learning accounts closed by prompts aimed at identifying, interpreting and deciding on utterances of mathematical naming and explaining. The transcribed accounts and utterances in WS2 were fictitious and those in WS1 reproduced data owned by the research team. The WS1 tasks functioned together to work on practices of naming and explaining in teaching talk that were responsive to challenges around linear equations that students in many classrooms face. This was also the case with the WS2 tasks, but for challenges around probability.

Figures 1 and 2 reproduce abbreviated versions in English of a WS1 task and a WS2 task, with a summary of the teaching and learning account. The task in Fig. 1 is centred on teaching and learning challenges in regard to algebra structure sense development (Rojano, 2022). The task in Fig. 2 is centred on challenges in regard to the use of the representativeness heuristic. A critical aspect was the potential for every task to prompt the teachers' noticing of the importance of mathematical naming and explaining in

Fig. 1 Abbreviated English version of a WS1 task

Secondary-school teaching and learning account

In two lessons on the algebra of equations aimed at discussing whether $x=5$ and $3x=15$ are the same equation, *same equation* was differently named by the two teachers as follows: *equal equation, equivalent equation, equation with everything identical or almost, comparable in solution, equation after applying some rules of transposition, same number or expression adding or multiplying both sides,* and *same line graph.* The relation between $x=5$ and $3x=15$ was also explained differently by these teachers to their students, as follows:

The value of x is not necessarily five in any equation, but it is for these two, so same equation.

They are identical to each other in the numerical solution.

Like five equals x same as x equals five. Different representation, same equation.

Let us think of three times five and three times x, and how the three is important here.

Fifteen divided by three is five, that is the key to start.

The two equations are equal because you can get the second one by simplifying the first.

You can compare the two equations by seeing that if you have the value of x in the first you can reason the value of x in the second.

Written prompts for group discussion

- How do these utterances of teaching talk support the learning of linear equations?
- Which utterances support algebra structure sense? Which ones rather promote unreasoned ways of manipulating notation and finding the solution?
- Propose your ways of naming *same equation* and of explaining $x=5$ in relation to $3x=15$ to support algebra structure sense.

Fig. 2 Abbreviated English version of a WS2 task

Secondary-school teaching and learning account

In a lesson aimed at discussing the conditions of "equally likely cases" and "equally likely" in the classical definition of probability as the ratio between the number of favourable cases and the number of all possible cases, the teacher proposed the following mathematical task: *A box contains 2 black, 2 white and 2 blue balls. You take 4 balls at random, one at a time. Each time a ball is picked, the colour is recorded, and the ball is put back in the box. If the first three balls are black, what colour is the fourth ball least likely to be?*

a) White b) Blue c) Black d) All colours are equally likely

In their responses, some students chose option *c*, and feedback about the incorrectness of the choice was mainly centred on the detailed description of the experiment.

Written prompts for group discussion

- Which mathematical names and explanations would you say to support the probabilistic reasoning of the students?
- Ask some other teachers in the workshop for the mathematical names and explanations that they would use. Do you agree with all their proposals of names and explanations, or would you recommend some changes?
- Make and justify improvement proposals for some of the explanations collected.
- Did you find, in responses of other teachers in the group, mathematical names regarding probability that you had not thought of at first? Which ones? How do these mathematical names support the resolution of the mathematical task?

content-specific teaching talk. Some tasks from the early pilots had been modified and a procedure to validate the new tasks was necessary. These new tasks were discussed in a seminar, and feedback was applied to improve the formulation of the accounts and prompts and to ensure a balanced distribution of our adaptations of the three layers in van Es and Sherin (2002). The refined tasks were piloted in interviews with secondary-school mathematics teachers who were asked to read and comment on responses for the prompts. The final version of every task would enact some focused seeing through prompts aimed at identifying, interpreting and deciding on lesson-situated mathematical naming and explaining.

In WS2, there was a task similar to the one summarised in Fig. 1 in which the teachers were given a sample of teaching utterances (though fictitious and regarding probability). In WS1, there was also a task similar to the one summarised in Fig. 2 in which the teachers were first invited to generate teaching utterances (though regarding linear equations) before reflecting on them. We exemplify two tasks with two different formats to illustrate the variety of ways in which the teachers were induced into identifying, interpreting and deciding on naming and explaining. The written versions of the tasks in WS1 and WS2 offered isolated utterances of teacher talk, as Figs. 1 and 2 show. Nonetheless, the teacher educator contextualised the exemplified utterances within moments of the corresponding teaching and learning account, including mentions of students' interventions. Given our sociocultural theoretical stances, we wanted the teachers to view the practices of naming and explaining in relation to classroom work and as involving other practices and issues of metacommunication. At the same time, we wanted them to keep the focus or at least some emphasis on practices of naming and explaining. For this, we decided to present the classroom situation orally. After having reminded the teachers that teaching talk is part of a communicative dialogue in many directions and with many agents, we presented the utterances limited to the naming and explaining of the classroom teacher in the writing provided. The tasks were performed orally and in writing. In both cases, the teachers could comment or allude to the information written on the paper sheet or talked by the teacher educator.

4.2 Data collection and analyses

The two workshops, WS1 and WS2, were audio-recorded, all audio-recorded data were transcribed verbatim with numerical codes assigned for subsequent management, and the written responses of the teachers to the task prompts were collected to back up, if necessary, analyses of group discussions. Confidentiality was assured by means of pseudonyms for the teachers and their schools. A hybrid process of qualitative methods of deductive and inductive thematic text analysis (Proudfoot, 2023) was then initiated on the textual data from the transcripts in the languages of the teachers, Catalan and Spanish. The first two authors independently coded the WS1 and WS2 transcripts and, when they met to share their coding and reconcile minor differences, some nuances in the data were raised, discussed and agreed on. Whereas the differences in noticing between the two groups of teachers was itself an important finding that we came across, we still wanted to examine the possibility of some commonalities, on which this paper is centred. The risks around forcing the codes and coding were then reduced by expanding the team of coders to include two researchers who had not participated in the workshops' design and conducting, and who were not close to the teachers.

A second round of deductive analyses began with on-line meetings over several months, in which the four authors examined the coding framework, consisting of the themes induced by the first two authors from the Group 1 and Group 2 data. We worked with narrative approaches to the themes, supported by one example of WS2 transcript for each. This strategy allowed the third and fourth authors to work as independent coders. These authors have strong expertise in qualitative research involving coding methods and know well the theoretical framing of the study. The first two authors independently returned to the data, whereas the other two authors independently worked to identify and revise the themes in their reading of WS1 and WS2 data. Independent coding work was followed by group discussions in the meetings, with attention to interpretation agreements that might confirm refined versions of the themes in WS1 and WS2. We finally agreed on three themes that characterise three common foci of the teachers' noticing in the two groups:

– *Missing practices of mathematical naming in own teaching talk*. The teachers noticed a lack of important mathematical naming practices in their own classroom teaching talk, and a subsequent loss of responsiveness to the students' content-specific learning.
– *Relative impact of mathematical explaining in teaching talk*. The teachers noticed some relative impact of mathematical explaining in teaching talk mainly due to features of students' cognition and to the epistemic complexity of the mathematical content.
– *Tensions around mathematical naming and explaining in teaching talk*. The teachers noticed mathematical naming and explaining in teaching talk as in tension with practices of small group work and students' autonomous and experiential learning.

The very final stage was for linking the themes to selected quotations. This involved re-reading the data transcripts to identify illustrative examples of the teachers' noticing during the task discussions with which to communicate the presence and meaning of each theme.

5 Results common to the two groups of teachers

We elaborate three responses to the question: *What do the groups of mathematics teachers notice during their discussion of tasks on mathematical naming and explaining in content-specific teaching talk?* The responses, in the form of themes that emerged in WS1 and WS2, are presented together with examples of the transcribed talk of teachers of Groups 1 and 2. The initial time for individual thinking and written responses to the tasks had created a context of support for noticing development, but participation in the discussion of the responses importantly allowed the teachers of these groups to notice nuances of mathematical naming and explaining in teaching talk unaddressed in the task prompts. The group discussions thus amplified and opened up the opportunities to develop some focused noticing on the content of the workshops, specifically in connection with the teachers' own teaching practice. Amador et al. (2023), in the context of a one-year video club, documented moves towards collective noticing. Our results show that the social construction of teacher noticing is a feature of noticing development that can be also documented in the context of one-day workshops.

5.1 Missing practices of mathematical naming in own teaching talk

The group discussions, in WS1 and WS2, of utterances of teaching talk including mathematical naming practices revealed a focus of the teachers' noticing on the significance of mathematical naming in the teaching of mathematics and, particularly, in their own content teaching. The two groups of teachers noticed mathematical naming as using mathematical vocabulary and using it correctly, as avoiding ambiguous pronouns in the interaction with the students, but also as distinct from mathematical explaining and having a function in relating and combining meanings communicated through different modes or sign systems. In this respect, the teachers noticed a lack of mathematical naming in their teaching talk, such as naming aimed at communicating the mathematical content in focus, connecting mathematical words with mathematical symbols and diagrams in the classroom discussions, or introducing the mathematical concepts and processes of the school curriculum related to linear equations or probability. In the same way as the teachers in Adler (2021), our two groups related the missing naming practices to a loss of responsiveness to the students' learning in their lessons.

The teachers in WS2, when thinking of their classroom teaching talk, noticed the correct use of mathematical names and practices of replacing words, such as possibility, which are not mathematical or can be used incorrectly in naming the concepts related to probability. At the same time, however, they noticed a lack of important naming practices in their teaching, such as those involved in using mathematical words and phrases to name differently the numerator in the fractional notation of probability as the number of favourable outcomes of an experiment or as outcomes of an experiment that are possible and favourable. The mathematical naming noticed as intended in their teaching talk was expected to function for the explicit communication of mathematically precise meanings (Otten et al., 2019). In the WS2 group discussion, one of the teachers stated the following:

> I try to use the expressions in the same way, with precision, and I take the opportunity to make explicit that in mathematics the words often have a precise meaning associated with them. I place special emphasis on the difference between possibility and probability and on what it means if we use the plural of these words.

The teachers in WS1, when thinking of their own classroom teaching talk, also distinguished naming from explaining and they actually noticed some mathematical naming practices functioning to close opportunities for explaining and reasoning (e.g., "because it is an equation"). They noticed missing practices of mathematical naming, specifically those involved in associating mathematical names with other mathematical names correctly, avoiding ambiguous pronouns (Longwe et al., 2022), and naming the mathematical symbolic notation of linear equations and parts of it with mathematical words (e.g., naming $x = 5$ as an equation, compared to reading aloud the symbols, "x equals five"). This result indicates some noticing of the criticality of mathematical naming in the own teaching, and of the learning opportunities missed for the students in the classroom. Two of the teachers addressed these issues in the WS1 group discussion as follows:

> In the classroom, I don't know exactly how I say it or how I explain it, but I want it to depend a bit on the students I have that year. I explain more if they are students who ask more … I explain ways of reasoning, not names… I don't think I comment on how we're going to read in words the equation that is written in symbols, or why we name it linear. It's not because we draw any line, because the line has to be straight, not a parabola, but this name... I don't explain the names.
>
> I was thinking that I can't always explain things like for example that x equals five is an equation, because in my classes we aren't always at that point. I mean that to explain this or to read the equation in different ways is perhaps for when they know a bit more … But yes, with symbols and that's it, I don't help much.

More generally, and in line with the contextualized presentation of the task utterances within classroom situations by the teacher educator in each workshop, when the teachers of Groups 1 and 2 discussed missing practices of mathematical naming in their teaching talk, they raised views of mathematical naming as happening in the communication with the classroom students. Naming was thus seen at least from two sides at once: the side of the teacher in teaching and the side of the students in their interacting with the teachers' naming (e.g., "If you don't say possible and favourable cases, they don't say it either"), the mathematical representations (e.g., "The equation that is written in symbols") and the teaching and learning account (e.g., "Problems of balls or problems of equally likely"). In this respect, mathematical naming was noticed in the group discussions within a third space (Williams & Ryan, 2013) originated in the school classroom, with the naming utterances of the teacher opening and following up communications or dialogues with the students and their own talk.

5.2 Relative impact of mathematical explaining in teaching talk

A result in Planas and Alfonso (2023) showed the Group 1 teachers' noticing of mathematical explaining in teaching talk as supporting classroom mathematical discussions and connecting teaching and learning by using students' contributions to voice content learning challenges. In the Group 2 discussions of mathematical explaining in teaching that was responsive to the students' learning, the voicing of learning challenges in teaching talk was also noticed and valued. However, a number of nuances were captured in how the two groups noticed the importance of mathematical explaining in teaching talk. The teachers in WS1 and WS2 discussed concerns regarding some relative impact of mathematical explaining on the students' learning, because of circumstances over which they had minimum control, such as the students' listening practices, preconceptions and beliefs, and because of the epistemic and cognitive complexity of linear equations and probability. Compared with the focus on mathematical naming in teaching and how it led to recognise the lack of some important naming practices in the own teaching talk, the focus on mathematical explaining was less supportive of reflection on the own teaching talk and the potential of teaching in general.

The teachers in WS1 and WS2 noticed students' listening as required for an impact of the mathematical explaining of the teacher. They raised cases of students with their head down or looking through the window as not necessarily implying experiences of challenges in the learning of linear equations or probability, but rather indicating experiences of listening to the teacher or to others in the classroom as not helpful. In noticing the students' listening as integral to the accomplishment of the function of mathematical explaining in teaching, the teachers from Group 1 noticed the significance of general talk moves of the teacher in this regard (Ingram et al., 2019), and the modelling for the students of the practice of listening to one another (O'Connor & Michaels, 2019). This is well illustrated with the following quote:

> We are presupposing that the students listen to us when we talk. But if they do not listen to us, how we talk to them perhaps does not matter that much … Not that what we say is unhelpful, but perhaps not so important … If we explain why the algebraic expression is an identity, but they are waiting to know which exercise they have to do, well, they don't listen to the explanation because maybe it is not practised to listen to the explanation or to listen to each other.

The teachers of the two groups also noticed the epistemic complexity of the algebra of equations and of probability, and the diversity of students' preconceptions, beliefs and moments of cognitive development and individual learning coexisting in any classroom. All these variables were associated with the difficult accomplishment of certain conditions for successful mathematical explaining in teaching. This focus of the noticing on conditions for mathematical explaining suggests the issue of the varying depth or quality of mathematical explanations (Evans et al., 2022), and the teacher work and pedagogic knowledge involved. Nonetheless, the teachers' attention to the problem of the difficult adaptation of mathematical explaining to the many demands of school mathematics did not turn into specific explaining proposals. This was so even though the task prompts were aimed at identifying, interpreting and deciding on utterances of mathematical explaining. The quotes below are from teachers who participated in discussions, with no specific explaining proposals, on the difficulty of explaining the equal sign and the probability concept.

> This is not easy to explain, because this is nothing less than algebra. We have explained to students for years that the equal sign goes with operations, such as five and three plus two, and now this equal sign is different. Probability is difficult to explain, because the students bring in preconceived ideas that are not very rigorous … Here the teacher explains, but you cannot expect the students to change their beliefs … It does not lead the students to overcome their misconceptions.

In a similar way to what was found for mathematical naming, when the teachers of Groups 1 and 2 discussed the impact of mathematical explaining, they raised views of mathematical explaining as happening in the communication with the classroom students. Responsiveness to mathematical explaining in teaching was thus noticed in relation to responsiveness towards the students' participation in mathematics. Explaining was also seen from two sides at once: the side of the teacher in teaching and the side of the students in their interacting with the teachers' explaining (e.g., "We can use some of their words like the left and the right sides of the equation in our explanations") and with the mathematical content (e.g., "Conceptual probability is more challenging and difficult to explain than empirical probability"). This noticing of mathematical explaining in teaching in a relationship of dependency with the circumstances of the students and the content of learning reminds us of the inner role of contingencies in the development of teachers' noticing (Mason, 2002).

5.3 Tensions around mathematical naming and explaining in teaching talk

One more focus of the groups' noticing included joint attention to mathematical naming and explaining in teaching talk. These practices were noticed as important for the development of mathematical discussions in the classroom and, at the same time, in tension with the enactment of small-group work in the learning of linear equations (WS1) and probability (WS2). Mathematical naming and explaining in teaching were thought of in relation to the classroom whole group, and time for them was noticed as detrimental to time for enabling students to practise their mathematical talk and to come up with their ideas in small-group work. Although situations of students working together on a task are a common feature of the mathematics classrooms in the country, the teachers did not consider mathematical naming and explaining in response to small-group work. Their mentions of students' work were tied to accounts of autonomous learning and initiatives with algebra tiles and weighing balances in WS1, and with material experiments and simulations in WS2, and separated from the mathematical talk of the teacher except for interventions aimed at posing questions.

A focus of the teachers' noticing was the students' mathematical talk in small-group work and how time of classroom teaching talk for mathematical naming and explaining –compared and contrasted with teaching talk for introducing manipulatives, giving instructions, asking questions, or starting a discussion– could be detrimental to or limit these practices. This result resonates with the teachers in Planas et al. (2023) and their seeing of mathematical explaining in teaching talk as close to the "spoon feeding" (p. 528) that equates with the teacher explaining while students listen passively, hence coming into tension with student-centred mathematical pedagogies. In a WS1 discussion on "letting the students name and explain by themselves by providing materials, not talk", one of the teachers contributed as follows:

> If students work and talk in small groups to solve problems, it's better not to talk much ... They have to talk and explain themselves ... We can give them time to think ... It's even better if we encourage them to use manipulatives such as algebra tiles for them to represent the equation to each other.

The quote above also shows one more focus of the teachers' noticing on forms of communication in mathematics other than talk, with emphasis on communication with material objects in mathematics learning (Pimm & Sinclair, 2009). Mathematics teaching talk was seen as not always necessary and sometimes interfering with students' mathematical practices of exploring and thinking with objects. In WS1, the teachers saw algebra tiles as useful for teaching linear equations and physical action on these tiles as useful for learning the concept. In WS2, the teachers discussed the use of materials and games for playing chance experiments, guessing outcomes and evaluating chances in small-group work, with teacher interventions mainly aimed at promoting students' exploration. The quotes below illustrate this emphasis and tensions around naming and explaining that are not sufficiently specific.

> I think that in both cases the expressions and explanations are rather unspecific… I think the use of materials and doing the experiment would improve the expressions and explanations. I think it's difficult to explain this topic without doing the experiment.
> I would bring to class a bag with the balls in the wording, and I would make each student pick one and return it to the bag, hoping that they would realise that they have an equal probability of picking a black one … I would bring the roulette to class, and they would do all three spins several times, insisting on the difference.

This result of the teachers' noticing supports the understanding of the mathematical naming and explaining of the classroom teacher as contingent on the interactions with the students and their talk, but also on the interactions with material tasks. The functions of mathematical naming and explaining are thus seen from three related sides at once: the side of the teacher, the side of the students and the side of the mathematical activity in the classroom. Like in van Es et al. (2017), the teachers' noticing in our study shows the important pedagogic practice of viewing teaching and teaching talk through critical lenses and competing priorities. Informed by their own teaching experiences and their sharing in the discussions, the two groups of teachers introduce reflections on teaching time that is responsive with the students' learning time, and on classroom talk that is responsive with forms of non-linguistic communication in the mathematical interaction of the students with materials.

6 Conclusions and prospective

We have presented noticing practices for two groups of teachers during their participation in workshops on mathematical naming and explaining in teaching talk. Our three results show three foci of the groups' noticing. A first focus is on the place of mathematical naming in own teaching talk, including the challenges of using mathematical words correctly (e.g., possibility, probability and plurals) and of connecting mathematical symbolic language (e.g., algebraic expressions, the probability fractional notation) and diagrams (e.g., graphs of linear equations, probability as area) with mathematical words. A second focus is on the relative impact of the teacher's mathematical explaining, including limitations due to the students' cognition (e.g., everyday non-mathematical notions of probability) and practices of listening and not-listening, and to the epistemic complexity of the mathematical content (e.g., the distinction between equation and identity or between empirical and theoretical probability). A third focus is on mathematical naming and explaining as practices in tension with some classroom practices (e.g., teaching talk that hinders small-group work) and forms of communication (e.g., teaching talk that hinders thinking and reasoning with objects). These results reveal that opportunities of noticing development were created in the context of the task-prompted discussions following the time for individual thinking and writing and individual noticing in the workshops. Our study hence contributes to mathematics education research that aims at understanding the facilitation and opening up of teacher noticing through participation in group discussions (e.g., Amador et al., 2023; Barnes & Solomon, 2013; Sjöblom et al., 2023). It also contributes to diversifying the foci of mathematics teacher noticing by addressing the pedagogic and content-specific role of mathematical naming and explaining in teaching talk (e.g., Adler et al., 2023; Longwe et al., 2022; Planas et al., 2023).

In subsections 5.1, 5.2 and 5.3, we have illustrated individualised accounts of transcribed talk of teachers engaged in practices of identifying, interpreting and making decisions with others. Although these accounts provide some evidence of what different teachers noticed individually about the specifics and nuances of mathematical naming and explaining and about more general issues regarding content-specific teaching and learning, the teachers' collective experience in the workshops and their thinking of the tasks together were fundamental. Even when the teachers in WS1 and WS2 introduced aspects of their individual teaching in their classrooms, these aspects were considered in the discussions with the other teachers and the teacher educator, and what was said was inevitably mediated by who the others were and what they had said. This approach to the social understanding of individualised data distinguishes our study from other studies on mathematics teachers' noticing that exclusively consider individual noticing in educational settings that would allow to consider notice development in group as well. By prioritising the data from the discussions, where the teachers made their noticing visible to one another so that noticing developed dialectically (Barnes & Solomon, 2013), we contribute to creating a more balanced research domain in terms of addressing individual and social aspects.

The teachers' noticing in the discussions developed dialectically in many respects. Practices of mathematical naming or explaining were not assessed in the group without considering other practices such as students' talk in classroom activity, which might imply the use of forms in teaching that are mathematically less precise or accurate. The teachers noticed the importance of the precise language of the mathematics register (Halliday, 1975) and, at the same time, they noticed the importance of relating mathematical naming and explaining to the mathematical languages of the students by, for example, borrowing some of the ways that students use to reason through talk in lesson discussions. In this regard, the mathematical naming and explaining of the teacher in the classroom was noticed as consisting of two different types of practices –guided by the precision of the mathematics register and the metacommunication of students' talk–, none of which seemed more significant than the other. We are currently discussing explicit ways of addressing mathematical naming and explaining as contingent on and supportive of the students' mathematical practices and talk, in connection with other discourse practices and forms of communication. Task prompts that directly allude to the teachers' classroom activity could support some balanced attention to the mathematics register and students' participation in mathematics. A plan is to incorporate workshops on mathematical naming and explaining in prospective teacher education programmes of our three universities. We encourage other researchers who are mathematics teacher educators with curricular responsibilities to consider the inclusion of this content as well.

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